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Herz Type Spaces and Their Applications

(赫 兹 型 空 间 及 其 应 用)

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Preface

The Herz spaces are a class of function spaces introduced by Herz in the study of absolutely convergent Fourier transforms in 1968. More recently, great attention was paid to the study on the Herz spaces since there are several remarkable works to push forward the study on the Herz spaces. Firstly, Chen-Lau and Garcia-Cuerva studied some Hardy spaces associated with nonhomogeneous Herz spaces with special indexes in $n = 1$ and $n > 1$ respectively. Secondly, Lu and Yang, the first two authors of this book, studied some Hardy spaces associated with homogeneous Herz spaces with special indexes. Finally, the complete theory for the case of general indexes was established by Lu and Yang in 1995. Precisely, Lu and Yang first established a decomposition characterization of the Herz spaces in terms of so-called central blocks. Then they established a decomposition characterization of Hardy spaces associated with Herz spaces (called Herz type Hardy spaces) in terms of central atoms both for homogeneous and nonhomogeneous case. Lu and Yang also established a decomposition theorem of Herz type Hardy spaces in terms of central molecules. These decomposition theorems greatly push forward the study on applications of Herz spaces and Herz type Hardy spaces. Three important applications are worth pointing out here. Firstly, certain Herz type Hardy space will be a proper substitute of Hardy space in the study of boundedness of oscillatory singular integral operators with polynomial phase since the boundedness of these operators fails on Hardy spaces. Secondly, some Herz spaces with weights are very useful in the study of boundedness of multipliers. Finally, the boundedness of sublinear operators on homogeneous Herz spaces can be regarded as a natural generalization of that on weighted Lebesgue spaces with power weights.

The whole book consists of six chapters. The definition of Herz spaces together with its basic properties will be briefly introduced in Chapter 1. It includes central block decomposition of Herz spaces. The contents in Chapter 2 are related the basic theory of Herz type Hardy spaces. They mainly involve central atom decomposition and central molecular decomposition of Herz type Hardy spaces. Chapter 3 is related to characterizations of wavelet transforms and ϕ-transforms of Herz type Hardy spaces in the sense of Frazier-Jawerth. The interpolation theory of Herz spaces and its applications will be discussed in Chapter 4. The content in the final two chapters is devoted to applications on boundedness of sublinear operators on Herz spaces and Herz type Hardy spaces.

The authors of this book would like to express their great thanks to all collaborators in the study of this subject for their contributions. Finally, this book is also in memory of Professor Cheng Minde and Professor Sun Yongsheng.

Shanzhen Lu
May, 2007

Contents

Symbols

- $\mathbb{R}^n$——the n-dimensional Euclidean space;
- S^{n-1}——the unit sphere in $\mathbb{R}^n$;
- $\mathbb{C}$——the set of all complex numbers;
- Δ——the open unit disc in $\mathbb{C}$;
- $\mathbb{Z}$——the set of all integers;
- $\mathbb{N}$——the set of all positive integers;
- $\mathbb{Z}_+$——$\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$;
- C——a constant which is independent of the main parameters, but may vary from line to line;
- c_j^k——the j-th constant appears in Chapter k, do not change in this chapter;
- $|E|$——for a measurable set $E \subset \mathbb{R}^n$, $|E|$ is the d-dimensional Lebesgue measure;
- χ_E——for a Lebesgue measurable set $E \subset \mathbb{R}^n$, χ_E is the characterization function of E;
- p'——for p with $1 \leqslant p \leqslant \infty$, p' is the dual exponent of p, namely, $1/p + 1/p' = 1$; for p with $0 < p < 1$, $p' = \infty$;
- $L^p(\mathbb{R}^n)$——the usual Lebesgue space with exponent p;
- $L^{p,\infty}(\mathbb{R}^n)$——the usual weak $L^p(\mathbb{R}^n)$ space;
- weight——by a weight w, we mean w is a nonnegative, measurable and locally integrable function;
- $L^p(\mathbb{R}^n, w)$——weighted $L^p(\mathbb{R}^n)$ space with weight w, namely,

$$L^p(\mathbb{R}^n, w) = \Big\{ f : f \text{ measurable on } \mathbb{R}^n \text{ and}$$
$$\|f\|_{L^p(\mathbb{R}^n, w)} = \Big(\int_{\mathbb{R}^n} |f(x)|^p w(x) \mathrm{d}x \Big)^{1/p} < \infty \Big\}.$$

- $[a]$——the integer part of the real number a;
- $\mathscr{S}(\mathbb{R}^n)$——the space of Schwartz functions;
- $\mathscr{D}(\mathbb{R}^n)$——$\mathscr{D}(\mathbb{R}^n) = \cap_{k=1}^{\infty}\mathcal{C}_0^k(\mathbb{R}^n)$, where $\mathcal{C}^k(\mathbb{R}^n)$ is the space of k times continuously differentiable complex-valued functions in $\mathbb{R}^n$;
- $\mathscr{S}'(\mathbb{R}^n)$——the dual space of $\mathscr{S}(\mathbb{R}^n)$;
- $\mathscr{D}'(\mathbb{R}^n)$——the dual space of $\mathscr{D}(\mathbb{R}^n)$;
- $\mathscr{P}_s(\mathbb{R}^n)$——the set of polynomial of degree s in $\mathbb{R}^n$;
- $\mathscr{P}(\mathbb{R}^n)$——the set of all polynomial on $\mathbb{R}^n$, namely, $\mathscr{P}(\mathbb{R}^n) = \cup_{s\geqslant 0}\mathscr{P}_s(\mathbb{R}^n)$;
- $A_p(\mathbb{R}^n)$——the weight function class of Muckhoupt. For $p \in (1,\, \infty)$, a weight w is said to be a $A_p(\mathbb{R}^n)$ weight if there exists a constant $C > 0$, such that for any cube Q,

$$\frac{1}{|Q|}\int_Q w(x)\,\mathrm{d}x\Big(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}(x)\,\mathrm{d}x\Big)^{p-1} \leqslant C;$$

  a weight w is said to be a $A_1(\mathbb{R}^n)$ weight if there exists a constant $C > 0$ such that for any cube Q,

$$\frac{1}{|Q|}\int_Q w(x)\,\mathrm{d}x \leqslant C\mathrm{essinf}_{x\in Q} w(x);$$

- $\mathcal{M}$——the standard Hardy-Littlewood maximal operator;
- $\mathcal{M}_r$——for some $r \in (0,\infty)$ and suitable function f, $\mathcal{M}_r f(x) = \big(\mathcal{M}(|f|^r)(x)\big)^{1/r}$;
- λB and λQ——for a ball $B = B(x,\, r)$, λB denote the ball $B(x,\, \lambda r)$; for a cube Q, λQ denote the cube having the same center as Q, but side length λ times that of Q;
- $\widehat{f}$——for a suitable function or a distribution f, $\widehat{f}$ denotes the Fourier transform;
- $f^{\vee}$——the inverse Fourier transform of f.

Chapter 1

Herz Spaces

The study of the Herz spaces can be dated back to the work of Beurling [Be]. In order to study certain convolution algebra, Beurling [Be] introduced a space A_q, which is the origin version of the Herz space of nonhomogeneous type, and is called Beurling algebras. Feichtinger [Fe] introduced another norm which is equivalent to the norm defined by Beurling. Herz [Hr] generalized this space of functions to further study the properties. These generalized spaces of functions are just the prototype of Herz spaces. In fact, Herz introduced the space K_q, which is known equivalent to the Herz space of homogeneous type with special exponent. Flett [Fl] gave a characterization of the Herz spaces K_q. In the 1990's, Lu and Yang introduced the homogeneous Herz space and non-homogeneous Herz space with general indices and established the block decomposition of the Herz spaces, from which they showed many interesting properties of these spaces.

This chapter is devoted to some preliminaries of Herz space and the Herz type spaces. We will give some definitions and notation, and establish some basic fact, such as the dual space, the block decomposition, for the Herz space.

1.1 Definitions and some basic facts

We begin with the necessary definitions. Let $B_k = \{x \in \mathbb{R}^n : |x| \leqslant 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\widetilde{\chi}_k = \chi_k$ if $k \in \mathbb{N}$ and $\widetilde{\chi}_0 = \chi_{B_0}$, where χ_{A_k} is the characteristic function of A_k.

Definition 1.1.1 Let $\alpha \in \mathbb{R}, 0 < p,\, q \leqslant \infty$.

(1) The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \Big\{ \sum_{k \in \mathbb{Z}} 2^{k\alpha p} \|f\chi_k\|^p_{L^q(\mathbb{R}^n)} \Big\}^{1/p}.$$

(2) The non-homogeneous Herz space $K_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$K_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\text{loc}}(\mathbb{R}^n) : \|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} = \Big\{ \sum_{k=0}^{\infty} 2^{k\alpha p} \|f\widetilde{\chi}_k\|^p_{L^q(\mathbb{R}^n)} \Big\}^{1/p}$$

(the usual modifications are made when $p = \infty$ and/or $q = \infty$).

Remark 1.1.1 In the sequel, we denote $K_q^{-n(1-1/q),\,1}(\mathbb{R}^n)$ with $1 < q < \infty$ by $A^q(\mathbb{R}^n)$, and call this space Beurling algebra.

Remark 1.1.2 The spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$ are quasi-Banach spaces and if $p, q \geqslant 1$, $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$ are Banach spaces.

Remark 1.1.3 It is obvious that $\dot{K}_q^{0,\,q}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ and for $\alpha \in \mathbb{R}$, $\dot{K}_q^{\alpha,\,q}(\mathbb{R}^n) = L^q(\mathbb{R}^n, |x|^{\alpha q})$, the Lebesgue space $L^q(\mathbb{R}^n)$ with power weight $|x|^{\alpha q}$. Therefore, the Herz spaces are natural generalization of the Lebesgue spaces with power weights.

Proposition 1.1.1 *Let $\alpha \in \mathbb{R}$, $0 < p, q \leqslant \infty$. The following inclusions are valid*:

(1) *if $p_1 \leqslant p_2$, then $\dot{K}_q^{\alpha,\,p_1}(\mathbb{R}^n) \subset \dot{K}_q^{\alpha,\,p_2}(\mathbb{R}^n)$ and $K_q^{\alpha,\,p_1}(\mathbb{R}^n) \subset K_q^{\alpha,\,p_2}(\mathbb{R}^n)$.*

(2) *if $\alpha_2 \leqslant \alpha_1$, then $K_q^{\alpha_1,\,p}(\mathbb{R}^n) \subset K_q^{\alpha_2,\,p}(\mathbb{R}^n)$.*

(3) *if $q_2 \leqslant q_1$, then*

$$\dot{K}_{q_1}^{\alpha,\,p}(\mathbb{R}^n) \subset \dot{K}_{q_2}^{\alpha-n(1/q_1-1/q_2),\,p}(\mathbb{R}^n), \quad K_{q_1}^{\alpha,\,p}(\mathbb{R}^n) \subset K_{q_2}^{\alpha-n(1/q_1-1/q_2),\,p}(\mathbb{R}^n).$$

This proposition can be proved by fairly simple computation. In fact, (1) is a consequence of the inequality

$$\Big(\sum_{k=1}^{\infty} |a_k|\Big)^r \leqslant \sum_{k=1}^{\infty} |a_k|^r, \quad \text{if } 0 < r < 1. \tag{1.1.1}$$

(2) and (3) can be deduced from the Hölder inequality directly.

About the relationship of $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$, we have

Proposition 1.1.2 *Let $0 < p, q \leqslant \infty$, $0 < \alpha < \infty$. Then*

$$K_q^{\alpha,\,p}(\mathbb{R}^n) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$$

and for $f \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$,

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} \approx \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} + \|f\|_{L^q(\mathbb{R}^n)}.$$

Proof Obviously, if $f \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, then

$$\begin{aligned}\|f\|^p_{K_q^{\alpha,p}(\mathbb{R}^n)} &= \Big(\int_{|x|\leqslant 1} |f(x)|^q \mathrm{d}x\Big)^{p/q} + \sum_{k=1}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)} \\ &\leqslant \|f\|^p_{L^q(\mathbb{R}^n)} + \|f\|^p_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.\end{aligned}$$

We claim that

$$\sum_{k=-\infty}^{0} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)} \leqslant C\|f\|^p_{L^q(|x|\leqslant 1)}.$$

In fact, an application of the inequality (1.1.1) gives us that

$$\begin{aligned}\sum_{k=-\infty}^{0} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)} &\leqslant \Big(\sum_{k=-\infty}^{0} 2^{k\alpha q}\|f\chi_k\|^q_{L^q(\mathbb{R}^n)}\Big)^{p/q} \\ &\leqslant \|f\|^p_{L^q(|x|\leqslant 1)}, \quad \text{if } q < p.\end{aligned}$$

On the other hand, by the Hölder inequality, we have

$$\begin{aligned}\sum_{k=-\infty}^{0} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)} &\leqslant \Big(\sum_{k=-\infty}^{0} 2^{k\alpha p(q/p)'}\Big)^{1/(q/p)'} \Big(\sum_{k=-\infty}^{0} \|f\chi_k\|^q_{L^q(\mathbb{R}^n)}\Big)^{p/q} \\ &\leqslant C\|f\|^p_{L^q(|x|\leqslant 1)}.\end{aligned}$$

Our claim then follows directly. Note that

$$\|f\|_{L^q(|x|\leqslant 1)} \leqslant \|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}.$$

Thus, to prove $f \in K_q^{\alpha,p}(\mathbb{R}^n)$ implies that $f \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, it suffices to prove that

$$\|f\|_{L^q(|x|>1)} \leqslant \|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}. \tag{1.1.2}$$

For the case that $0 < q < p$, it follows from the Hölder inequality that

$$\begin{aligned}\|f\|^q_{L^q(|x|>1)} &= \sum_{k=1}^{\infty} \|f\chi_k\|^q_{L^q(\mathbb{R}^n)} \\ &\leqslant \Big(\sum_{k=1}^{\infty} 2^{-k\alpha q(p/q)'}\Big)^{1/(p/q)'} \Big(\sum_{k=1}^{\infty} 2^{k\alpha p}\|f\chi_k\|^{p/q}_{L^q(\mathbb{R}^n)}\Big)^{q/p} \\ &\leqslant C\|f\|^q_{K_q^{\alpha,p}(\mathbb{R}^n)}.\end{aligned}$$

For the case that $p \leqslant q$, another application of (1.1.1) yields

$$\|f\|^q_{L^q(|x|>1)} \leqslant \Big(\sum_{k=1}^{\infty} \|f\chi_k\|^p_{L^q(\mathbb{R}^n)}\Big)^{q/p} \leqslant \|f\|^q_{K_q^{\alpha,p}(\mathbb{R}^n)}.$$

This completes the proof of Proposition 1.1.2. □

Proposition 1.1.3 *Let* $0 < q_1 \leqslant q < \infty$, $-n/q < \alpha < n(1-1/q)$. *Then for any* p *with* $0 < p < \infty$,

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) \subset L^{q_1}_{\mathrm{loc}}(\mathbb{R}^n).$$

Proof It suffices to prove that $f \in L^{q_1}(B_1)$. A trivial computation shows that

$$\begin{aligned}\|f\|^{q_1}_{L^{q_1}(\mathbb{R}^n)} &= \sum_{k=-\infty}^{0} \|f\chi_k\|^{q_1}_{L^{q_1}(\mathbb{R}^n)}\\ &\leqslant \sum_{k=-\infty}^{0} \|f\chi_k\|^{q_1}_{L^q(\mathbb{R}^n)} 2^{kn(1-q_1/q)}\\ &\leqslant C\|f\|^{q_1}_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \sum_{k=-\infty}^{0} 2^{k(n(1-q_1/q)-\alpha q_1)}\\ &\leqslant C\|f\|^{q_1}_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)},\end{aligned}$$

since $n(1-q_1/q) - \alpha q_1 > 0$. □

Proposition 1.1.4 (*the Hölder inequality in Herz spaces*) *If* $0 < p_i$, $q_i \leqslant \infty$, $-\infty < \alpha_i < \infty$, $i = 1, 2$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/q_1 + 1/q_2$ *and* $\alpha = \alpha_1 + \alpha_2$, *then*

$$\|fg\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \leqslant \|f\|_{\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)} \|g\|_{\dot{K}_{q_2}^{\alpha_2,p_2}}.$$

The proof of Proposition 1.1.4 is an exercise involving the Hölder inequality.

At the end of this section, we give the characterization of the space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ for some special exponent, in terms of the Littlewood-Paley operators. Let ψ is integrable on $\mathbb{R}^n$ and

(i) $\displaystyle\int_{\mathbb{R}^n} \psi(x)\,\mathrm{d}x = 0.$

(ii) $|\psi(x)| \leqslant C(1+|x|)^{-(n+\alpha)}$, for some $\alpha > 0$.

(iii) $\displaystyle\int_{\mathbb{R}^n} |\psi(x+y) - \psi(x)|\,\mathrm{d}x \leqslant C|y|^{\gamma}$, for all $y \in \mathbb{R}^n$, where $\gamma > 0$ is a fixed constant. Denote $\psi_t(x) = t^{-n}\psi(x/t)$ with $t > 0$ and $x \in \mathbb{R}^n$. For f in $L^2(\mathbb{R}^n)$ with compact support, the Littlewood-Paley g-function of f is defined by

$$g(f)(x) = \left\{ \int_0^\infty |f * \psi_t(x)|^2 \frac{\mathrm{d}t}{t} \right\}^{1/2}; \tag{1.1.3}$$

the Lusin area function of f is defined by

$$S_{\psi,a}(f)(x) = \Big(\frac{1}{a^n |B_0|} \int_{\Gamma_a(x)} |f * \psi_t(x)|^2 t^{-n}\, \mathrm{d}y \frac{\mathrm{d}t}{t} \Big)^{1/2}, \tag{1.1.4}$$

where $\Gamma_a(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : \ |x-y| < at\}$; and the Littlewood-Paley g^*_λ-function of f is defined by

$$g^*_{\psi,\lambda}(f)(x) = \left(\int_0^\infty \int_{\mathbb{R}^n} \frac{|f * \psi_t(y)|^2}{\Big(1 + \dfrac{|x-y|}{t}\Big)^{2\lambda}} t^{-n}\, \mathrm{d}y \frac{\mathrm{d}t}{t} \right)^{1/2}. \tag{1.1.5}$$

It is well known that[1], all those functions have $L^p(\mathbb{R}^n)$ norms equivalent to $\|f\|_{L^p(\mathbb{R}^n)}$ when f in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. For these operators on Herz spaces, we have[2]

Theorem 1.1.1 *Let* $1 \leqslant p \leqslant \infty,\ 1 < q < \infty,\ -n/q < \alpha < n(1-1/q)$ *and* ψ *satisfies* (i), (ii) *and* (iii) *above. If* $a > 0$ *and* $\lambda > 3n/2$, *then the following statements are equivalent*:

(i) $f \in \dot{K}^{\alpha,p}_q(\mathbb{R}^n)$.

(ii) $g_\psi(f) \in \dot{K}^{\alpha,p}_q(\mathbb{R}^n)$.

(iii) $S_{\psi,a}(f) \in \dot{K}^{\alpha,p}_q(\mathbb{R}^n)$.

(iv) $g^{\psi}_{\psi,\lambda}(f) \in \dot{K}^{\alpha,p}_q(\mathbb{R}^n)$.

There is a similar result for $K^{\alpha,q}_p(\mathbb{R}^n)$.

1.2 Dual spaces of the Herz spaces

This section is devoted to the dual spaces for Herz spaces as given in Definition 1.1.1. We will consider this problem in a more general setting.

Definition 1.2.1 Let $\alpha \in \mathbb{R}$ and $0 < p \leqslant \infty$. $\dot{\ell}^\alpha_p$ is the space of all real-valued sequences $a = \{a_j\}_{j=-\infty}^\infty$ such that

$$\|a\|_{\dot{\ell}^\alpha_p} = \Big\{ \sum_{j\in\mathbb{Z}} 2^{j\alpha} |a_j|^p \Big\}^{1/p} < \infty.$$

---

1 See [To], Chapt.12.

2 We will prove this in Section 5.4.

When $p=\infty$, we write $\|a\|_{\dot{\ell}^\alpha_\infty} = \sum_{j\in\mathbb{Z}} 2^{\alpha j}|a_j|$.

Definition 1.2.2 Let $\alpha \in \mathbb{R}$ and $0 < p \leqslant \infty$, $\dot{\mathcal{A}} = \{A_j\}_{j=-\infty}^{\infty}$ and $\mathcal{A} = \{A_j\}_{j=0}^{\infty}$ be two sequences of Banach spaces. We define

$$\dot{\ell}^\alpha_p(\dot{\mathcal{A}}) = \Big\{a|a=\{a_j\}_{j=-\infty}^{\infty}, a_j \in A_j \text{ and } \Big(\sum_{j=-\infty}^{\infty}(2^{j\alpha}\|a_j\|_{A_j})^p\Big)^{1/p} < \infty\Big\}$$

and

$$\ell^\alpha_p(\mathcal{A}) = \Big\{a|a=\{a_j\}_{j=0}^{\infty}, a_j \in A_j \text{ and } \Big(\sum_{j=0}^{\infty}(2^{j\alpha}\|a_j\|_{A_j})^p\Big)^{1/p} < \infty\Big\},$$

where the usual modifications are made when $p=\infty$. Moreover,

$$\dot{C}^\alpha_0(\dot{\mathcal{A}}) = \{a|a \in \dot{\ell}^\alpha_\infty(\dot{\mathcal{A}}) \text{ and } 2^{j\alpha}\|a_j\|_{A_j} \to 0 \text{ as } |j| \to \infty\}$$

and

$$C^\alpha_0(\mathcal{A}) = \{a|a \in \ell^\alpha_\infty(\mathcal{A}) \text{ and } 2^{j\alpha}\|a_j\|_{A_j} \to 0 \text{ as } j \to \infty\}.$$

For a normed linear space $\mathscr{X}$, let $\mathscr{X}^*$ be the dual space of $\mathscr{X}$. Set $\dot{\mathcal{A}}^* = \{A_j^*\}_{j=-\infty}^{\infty}$ and $\mathcal{A}^* = \{A_j^*\}_{j=0}^{\infty}$. If $0 < p \leqslant \infty$ and $1/p + 1/p' = 1 (p' = \infty$, if $0 < p \leqslant 1)$, then, for $f = \{f_j\}_{j=-\infty}^{\infty} \in \dot{\ell}^{-\alpha}_{p'}(\dot{\mathcal{A}}^*)$ and $a = \{a_j\}_{j=-\infty}^{\infty} \in \dot{\ell}^\alpha_p(\dot{\mathcal{A}})$, we define

$$f(a) = \sum_{j=-\infty}^{\infty} f_j(a_j), \tag{1.2.1}$$

which is a linear continuous functional over $\dot{\ell}^\alpha_p(\dot{\mathcal{A}})$ and satisfies

$$\|f\|_{(\dot{\ell}^\alpha_p(\dot{\mathcal{A}}))^*} \leqslant \|f\|_{\dot{\ell}^{-\alpha}_{p'}(\dot{\mathcal{A}}^*)}. \tag{1.2.2}$$

Similar results hold for the spaces $\ell^\alpha_p(\mathcal{A})$ and $\ell^{-\alpha}_{p'}(\mathcal{A}^*)$.

Theorem 1.2.1 *Let $\dot{\mathcal{A}}, \mathcal{A}$ be as above, $\alpha \in \mathbb{R}, 0 < p < \infty$ and $1/p+1/p' = 1$, where $p' = \infty$ if $0 < p \leqslant 1$. Then $\Big(\dot{\ell}^\alpha_p(\dot{\mathcal{A}})\Big)^* = \dot{\ell}^{-\alpha}_{p'}(\dot{\mathcal{A}}^*)$, $\Big(\dot{C}^\alpha_0(\dot{\mathcal{A}})\Big)^* = \dot{\ell}^{-\alpha}_1(\dot{\mathcal{A}}^*)$, $\Big(\ell^\alpha_p(\mathcal{A})\Big)^* = \ell^{-\alpha}_{p'}(\mathcal{A}^*)$, and $\Big(C^\alpha_0(\mathcal{A})\Big)^* = \ell^{-\alpha}_1(\mathcal{A}^*)$.*

Proof We only prove the theorem for the spaces $\dot{\ell}_p^\alpha(\dot{\mathcal{A}})$. We start with the case $1 < p < \infty$. Let $f \in \left(\dot{\ell}_p^\alpha(\dot{\mathcal{A}})\right)^*$. If $a_j \in A_j, j \in \mathbb{Z}$, we define $\delta_j a_j = \{\delta_{jk} a_k\}_{k=-\infty}^{\infty}$, where

$$\delta_{jk} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $f_j(a_j) = f(\delta_j a_j)$ is a linear functional over A_j and

$$\sum_{j=-N}^{N} f_j(a_j) = f\Big(\sum_{j=-N}^{N} \delta_j a_j\Big).$$

Now, we choose $a_j' \in A_j$ with $\|a_j'\|_{A_j} = 2^{-j\alpha}$ such that $f_j(a_j')$ is real and

$$f_j(a_j') \geqslant 2^{-j\alpha}\|f_j\|_{A_j^*} - \varepsilon_j,$$

where $\varepsilon_j > 0$ are given numbers. We let $a_j = (2^{-j\alpha}\|f_j\|_{A_j^*})^{p'-1} a_j'$. For given $\varepsilon > 0$, we can choose $\varepsilon_j > 0$ such that

$$f_j(a_j) + \varepsilon 2^{-|j|} \geqslant 2^{-j\alpha p'}\|f_j\|_{A_j^*}^{p'} = 2^{j\alpha p}\|a_j\|_{A_j}^p.$$

Thus,

$$\begin{aligned}\sum_{j=-N}^{N} 2^{-j\alpha p'}\|f_j\|_{A_j^*}^{p'} &\leqslant 4\varepsilon + f\Big(\sum_{j=-N}^{N} \delta_j a_j\Big) \\ &\leqslant 4\varepsilon + \|f\|_{(\dot{\ell}_p^\alpha(\dot{\mathcal{A}}))^*}\Big(\sum_{j=-N}^{N} 2^{j\alpha p}\|a_j\|_{A_j}^p\Big)^{1/p}.\end{aligned}$$

Letting $\varepsilon \to 0$ and then $N \to \infty$, we obtain

$$\|\{f_j\}_{j=-\infty}^{\infty}\|_{\dot{\ell}_{p'}^{-\alpha}(\dot{\mathcal{A}}^*)} \leqslant \|f\|_{(\dot{\ell}_p^\alpha(\dot{\mathcal{A}}))^*}.$$

In addition, (1.2.1) and (1.2.2) indicate that we can construct a functional over $\dot{\ell}_p^\alpha(\dot{\mathcal{A}})$ by using $\{f_j\}_{j=-\infty}^{\infty} \in \dot{\ell}_{p'}^{-\alpha}(\dot{\mathcal{A}}^*)$. Note that this functional coincides with f for all elements of the form $\sum_{j=-N}^{N} \delta_j a_j$ and these elements are dense in $\dot{\ell}_p^\alpha(\dot{\mathcal{A}})$; hence, this functional coincides with f on the whole space $\dot{\ell}_p^\alpha(\dot{\mathcal{A}})$. From this and (1.2.1)$\sim$(1.2.2), Theorem 1.2.1 follows for the case $1 < p < \infty$. For $0 < p \leqslant 1$ choose $a_j = a_j'$ and $\varepsilon_j = \varepsilon$. For the case $\dot{C}_0^\alpha(\dot{\mathcal{A}})$, choose $a_j = a_j'$ and $\varepsilon_j = 2^{-|j|}\varepsilon$. $\square$

Now, let $f_k(x) = f(x)\chi_k(x)$ and $\widetilde{f}_k(x) = f(x)\widetilde{\chi}_k(x)$, then we have

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f_k\|_{L^q(\bar{A}_k)} \Big\}^{1/p} = \|\{f_k\}_{k=-\infty}^{\infty}\|_{\dot{\ell}_p^\alpha(\dot{\mathcal{A}})}$$

and

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} = \|\{\widetilde{f}_k\}_{k=0}^{\infty}\|_{\ell_p^\alpha(\mathcal{A}')},$$

where $\dot{\mathcal{A}} = \{L^q(\bar{A}_k)\}_{k=-\infty}^{\infty}$, $\mathcal{A}' = \{L^q(\bar{B}(0,1)), \{L^q(\bar{A}_k)\}_{k=1}^{\infty}\}$. From this and using Theorem 1.2.1, we easily obtain the following corollary. Before we state it, we need more definitions.

Definition 1.2.3 Let $\alpha \in \mathbb{R}$ and $0 < q \leqslant \infty$.

(1) $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ is said to belong to the space $(\dot{K}_q^{\alpha,\infty})_0(\mathbb{R}^n)$ if $f \in \dot{K}_q^{\alpha,\infty}(\mathbb{R}^n)$ and $2^{k\alpha}\|f\chi_k\|_{L^q(\mathbb{R}^n)} \to 0$, as $|k| \to \infty$.

(2) $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ is said to belong to the space $(K_q^{\alpha,\infty})_0(\mathbb{R}^n)$ if $f \in K_q^{\alpha,\infty}(\mathbb{R}^n)$ and $2^{k\alpha}\|f\widetilde{\chi}_k\|_{L^q(\mathbb{R}^n)} \to 0$, as $k \to \infty$.

Now, we can state our corollary as follows.

Corollary 1.2.1 *Let $\alpha \in \mathbb{R}, 0 < p < \infty, 1 \leqslant q < \infty$. Then*

$$\Big(\dot{K}_q^{\alpha,p}(\mathbb{R}^n)\Big)^* = \dot{K}_{q'}^{-\alpha,p'}(\mathbb{R}^n), \quad \Big(K_q^{\alpha,p}(\mathbb{R}^n)\Big)^* = K_{q'}^{-\alpha,p'}(\mathbb{R}^n)$$

and

$$\Big((\dot{K}_q^{-\alpha,\infty})_0(\mathbb{R}^n)\Big)^* = \dot{K}_{q'}^{\alpha,1}(\mathbb{R}^n), \quad \Big((K_q^{-\alpha,\infty})_0(\mathbb{R}^n)\Big)^* = K_{q'}^{\alpha,1}(\mathbb{R}^n)$$

(*recall that $p' = \infty$ if $0 < p \leqslant 1$*).

Notice that if $1 \leqslant p \leqslant \infty$ and $1 \leqslant q < \infty$, then the spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$ are Banach spaces. By the closed-graph theorem, we easily get the following corollary.

Corollary 1.2.2 *Let $\alpha \in \mathbb{R}, 1 \leqslant p, q < \infty$ and $1/q + 1/q' = 1 = 1/p + 1/p'$. Then $f \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ if and only if*

$$\Big| \int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x \Big| < \infty$$

for every $g \in \dot{K}_{q'}^{-\alpha,p'}(\mathbb{R}^n)$ and, in this case,

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \sup \Big\{ \Big| \int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x \Big| : \|g\|_{\dot{K}_{q'}^{-\alpha,p'}(\mathbb{R}^n)} \leqslant 1 \Big\}.$$

There exists a similar result for the non-homogeneous space $K_q^{\alpha,p}(\mathbb{R}^n)$.

1.3 Block decompositions for the Herz spaces

As well-known, one of the most significant progress in harmonic analysis in the last century is the development of the building block decomposition for some important function space [LTaW]. With the aid of the building block decomposition, the boundedness for some classical operators on these function spaces can be reduced to establishing the uniformly estimates for the operators on the building blocks. In this section, we will consider the decomposition of $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$. We begin with the notation of central block.

Definition 1.3.1 Let $0 < \alpha < \infty$, $1 \leqslant q < \infty$.

(i) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha,\, q)$-block if

(1) $\operatorname{supp} a \subset B(0,\, r)$ for some ball B centered at the origin and having radius r.

(2) $\|a\|_{L^q(\mathbb{R}^n)} \leqslant Cr^{-\alpha}$.

(ii) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha,\, q)$-block of restrict type if

(1) $\operatorname{supp} a \subset B(0,\, r)$ for some $r \geqslant 1$.

(2) $\|b\|_{L^q(\mathbb{R}^n)} \leqslant r^{-\alpha}$.

If $r = 2^k$ for some $k \in \mathbb{Z}$ in Definition 1.3.1, then the corresponding central block is called a dyadic central block.

The following decomposition theorem shows that the central blocks are the "buliding block" of the Herz spaces.

Theorem 1.3.1 *Let $0 < \alpha < \infty$, $0 < p < \infty$ and $1 \leqslant q < \infty$. The following two statements are equivalent:*

(i) $f \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.

(ii) *f can be represented by*

$$f(x) = \sum_{k\in\mathbb{Z}} \lambda_k b_k(x), \tag{1.3.1}$$

where each b_k is a dyadic central $(\alpha,\, q)$-block with support contained in B_k and $\sum_k |\lambda_k|^p < \infty$.

Proof We first prove (i) implies (ii). For $f \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, write

$$\begin{aligned} f(x) &= \sum_{k\in\mathbb{Z}} f(x)\chi_k(x) \\ &= \sum_{k\in\mathbb{Z}} |B_k|^{\alpha/n} \|f\chi_k\|_{L^q(\mathbb{R}^n)} \cdot \frac{f(x)\chi_k(x)}{|B_k|^{\alpha/n}\|f\chi_k\|_{L^q(\mathbb{R}^n)}} \end{aligned}$$

$$=\sum_{k\in\mathbb{Z}}\lambda_k b_k(x),$$

where

$$\lambda_k=|B_k|^{\alpha/n}\|f\chi_k\|_{L^q(\mathbb{R}^n)} \text{ and } b_k(x)=\frac{f(x)\chi_k(x)}{|B_k|^{\alpha/n}\|f\chi_k\|_{L^q(\mathbb{R}^n)}}.$$

It is obvious that $\operatorname{supp} b_k\subset B_k$ and $\|b_k\|_{L^q(\mathbb{R}^n)}=|B_k|^{-\alpha/n}$. Thus, each b_k is a dyadic central $(\alpha,\, q)$-block with the support B_k and

$$\sum_{k\in\mathbb{Z}}|\lambda_k|^p=\sum_{k\in\mathbb{Z}}|B_k|^{\alpha p/n}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)}=\|f\|^p_{\dot K^{\alpha,p}_q(\mathbb{R}^n)}<\infty.$$

Now we prove (ii) implies (i). Let $f(x)=\sum_{k\in\mathbb{Z}}\lambda_k b_k(x)$ be a decomposition of f which satisfies the hypothesis (ii) of Theorem 1.3.1. For each $j\in\mathbb{Z}$, it is readily to see that

$$\|f\chi_j\|_{L^q(\mathbb{R}^n)}\leqslant\sum_{k\geqslant j}|\lambda_k|\|b_k\|_{L^q(\mathbb{R}^n)}. \tag{1.3.2}$$

Now we consider two cases for the index p.

Case I. $0<p\leqslant 1$. From (1.3.2) it follows that

$$\begin{aligned}\|f\|^p_{\dot K^{\alpha,p}_q(\mathbb{R}^n)}&=\sum_{k\in\mathbb{Z}}2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)}\\&\leqslant\sum_{k\in\mathbb{Z}}2^{\alpha kp}\Big(\sum_{j\geqslant k}|\lambda_j|^p\|b_j\|^p_{L^q(\mathbb{R}^n)}\Big)\\&\leqslant\sum_{k\in\mathbb{Z}}2^{\alpha kp}\Big(\sum_{j\geqslant k}|\lambda_j|^p2^{-\alpha jp}\Big)\\&=\sum_{k\in\mathbb{Z}}|\lambda_k|^p\sum_{j\leqslant k}2^{\alpha(j-k)p}\leqslant C\sum_{k\in\mathbb{Z}}|\lambda_k|^p.\end{aligned}$$

Case II. $1<p<\infty$, again by (1.3.2) and the Hölder inequality,

$$\begin{aligned}\|f\chi_j\|_{L^q(\mathbb{R}^n)}&\leqslant\sum_{k\geqslant j}|\lambda_k|\|b_k\|^{1/2}_{L^q(\mathbb{R}^n)}\|b_k\|^{1/2}_{L^q(\mathbb{R}^n)}\\&\leqslant\Big(\sum_{k\geqslant j}|\lambda_k|^p\|b_k\|^{p/2}_{L^q(\mathbb{R}^n)}\Big)^{1/p}\Big(\sum_{k\geqslant j}\|b_k\|^{p'/2}_{L^q(\mathbb{R}^n)}\Big)^{1/p'}\\&\leqslant\Big(\sum_{k\geqslant j}|\lambda_k|^p2^{-\alpha kp/2}\Big)^{1/p}\Big(\sum_{k\geqslant j}2^{-\alpha kp'/2}\Big)^{1/p'}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|f\|^p_{\dot K^{\alpha,p}_q(\mathbb{R}^n)} &\leqslant C\sum_{j\in\mathbb{Z}}2^{\alpha jp}\Big(\sum_{k\geqslant j}|\lambda_k|^p 2^{-\alpha kp/2}\Big)\Big(\sum_{k\geqslant j}2^{-\alpha kp'/2}\Big)^{p/p'}\\ &\leqslant C\sum_{k\in\mathbb{Z}}|\lambda_k|^p\sum_{j\leqslant k}2^{\alpha(j-k)p/2}\\ &\leqslant C\sum_{k\in\mathbb{Z}}|\lambda_k|^p.\end{aligned}$$

This leads to that $f\in\dot K^{\alpha,p}_q(\mathbb{R}^n)$ and then completes the proof of Theorem 1.3.1 □

Remark 1.3.1 From the proof of Theorem 1.3.1, it is easy to see that if $f\in\dot K^{\alpha,p}_q(\mathbb{R}^n)$ and $f(x)=\sum\limits_{k\in\mathbb{Z}}\lambda_k b_k(x)$ be a central (α,q)-block decomposition, then

$$\|f\|_{\dot K^{\alpha,p}_q(\mathbb{R}^n)}\approx\Big(\sum_{k\in\mathbb{Z}}|\lambda_k|^p\Big)^{1/p}.$$

By an argument similar to the proof of Theorem 1.3.1, we can obtain the decomposition characterizations of the non-homogeneous Herz spaces as follows.

Theorem 1.3.2 *Let* $0<\alpha<\infty$, $0<p<\infty$ *and* $1\leqslant q<\infty$. *The following two statements are equivalent*:

(i) $f\in K^{\alpha,p}_q(\mathbb{R}^n)$.

(ii) f *can be represented by*

$$f(x)=\sum_{k=0}^{\infty}\lambda_k b_k(x), \tag{1.3.3}$$

where each b_k *is a dyadic central* (α,q)*-block of restrict type with support contained in* B_k *and* $\sum\limits_{k\geqslant0}|\lambda_k|^p<\infty$.

Moreover, the norms $\|f\|_{K^{\alpha,p}_q(\mathbb{R}^n)}$ *and* $\inf\Big(\sum\limits_{k\geqslant0}|\lambda_k|^p\Big)^{1/p}$ *are equivalent, where the infumum is taken all over all decompositions of* f *as in* (1.3.3).

Remark 1.3.2 Theorem 1.3.1 and Theorem 1.3.2 are also true when the Lebesgue measure is replaced by some suitable weights. To be precise, we first give the definition of weighted Herz spaces.

Definition 1.3.2 Let $\alpha \in \mathbb{R}$, p, $q \in (0, \infty)$ and ω_1, ω_2 be two weights on $\mathbb{R}^n$.

(1) The weighted homogeneous Herz space $\dot{K}_q^{\alpha, p}(\mathbb{R}^n;\, \omega_1,\, \omega_2)$ is defined by

$$\dot{K}_q^{\alpha, p}(\mathbb{R}^n;\, \omega_1,\, \omega_2) = \{f \in L_{\mathrm{loc}}^q(\mathbb{R}^n \setminus \{0\}, \omega_2) : \|f\|_{\dot{K}_q^{\alpha, p}(\mathbb{R}^n;\, \omega_1, \omega_2)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha, p}(\mathbb{R}^n;\, \omega_1, \omega_2)} = \Big\{ \sum_{k \in \mathbb{Z}} (\omega_1(B_k))^{\alpha p/n} \|f\chi_k\|_{L^q(\mathbb{R}^n,\, \omega_2)}^p \Big\}^{1/p}.$$

(2) The weighted non-homogeneous Herz space $K_q^{\alpha, p}(\mathbb{R}^n;\, \omega_1,\, \omega_2)$ is defined by

$$K_q^{\alpha, p}(\mathbb{R}^n;\, \omega_1,\, \omega_2) = \{f \in L_{\mathrm{loc}}^q(\mathbb{R}^n,\, \omega_2) : \|f\|_{K_q^{\alpha, p}(\mathbb{R}^n;\, \omega_1, \omega_2)} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha, p}(\mathbb{R}^n;\, \omega_1, \omega_2)} = \Big\{ \sum_{k=0}^{\infty} (\omega_1(B_k))^{\alpha p/n} \|f\widetilde{\chi}_k\|_{L^q(\mathbb{R}^n,\, \omega_2)}^p \Big\}^{1/p}$$

(the usual modifications are made when $p = \infty$ and/or $q = \infty$).

Definition 1.3.3 Let $0 < \alpha < \infty$, $1 \leqslant q < \infty$, ω_1, ω_2 are two weights on $\mathbb{R}^n$.

(i) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha,\, q;\, \omega_1,\, \omega_2)$-block if

(1) $\operatorname{supp} a \subset B(0,\, r)$ for some ball B centered at the origin and having radius r.

(2) $\|a\|_{L^q(\mathbb{R}^n, \omega_2)} \leqslant \big(\omega_1(B(0,\, r))\big)^{-\alpha/n}$.

(ii) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha,\, q;\, \omega_1,\, \omega_2)$-block of restrict type if

(1) $\operatorname{supp} a \subset B(0,\, r)$ for some $r \geqslant 1$.

(2) $\|b\|_{L^q(\mathbb{R}^n, \omega_2)} \leqslant \big(\omega_1(B(0,\, r))\big)^{-\alpha/n}$.

Parallel to the proof of Theorem 1.3.1, we can get

Theorem 1.3.3 *Let $0 < \alpha < \infty$, $0 < p < \infty$ and $1 \leqslant q < \infty$, $\omega \in A_1(\mathbb{R}^n)$ and ω_2 be a weight on $\mathbb{R}^n$. The following two statements are equivalent*:

(i) $f \in \dot{K}_q^{\alpha, p}(\mathbb{R}^n;\, \omega_1,\, \omega_2)$.

(ii) *f can be represented by*

$$f(x) = \sum_{k \in \mathbb{Z}} \lambda_k b_k(x),$$

where each b_k is a dyadic central $(\alpha,\, q;\, \omega_1,\, \omega_2)$-block with support contained in B_k and $\sum_k |\lambda_k|^p < \infty$.

Theorem 1.3.4 *Let $0 < \alpha < \infty$, $0 < p < \infty$ and $1 \leqslant q < \infty$, $\omega \in A_1(\mathbb{R}^n)$ and ω_2 be a weight on $\mathbb{R}^n$. The following two statements are equivalent*:

(i) $f \in K_q^{\alpha,\,p}(\mathbb{R}^n;\, \omega_1,\, \omega_2)$.

(ii) *f can be represented by*

$$f(x) = \sum_{k=0}^{\infty} \lambda_k b_k(x),$$

where each b_k is a dyadic central $(\alpha,\, q;\, \omega_1,\, \omega_2)$-block of restrict type with with support contained in B_k and $\sum_{k\geqslant 0} |\lambda_k|^p < \infty$.

1.4 The weak Herz spaces

As well-known, many classical operators, such as the Calderón-Zygmund operators, the Ricci-Stein oscillatory singular integral operators, are not bounded on $L^1(\mathbb{R}^n)$, they indeed map $L^1(\mathbb{R}^n)$ to $L^{1,\,\infty}(\mathbb{R}^n)$. So the space $L^{p,\,\infty}(\mathbb{R}^n)$ are very important in operator theory. It is natural to ask whether we can introduce some weak Herz spaces which are the analog of $L^{p,\,\infty}(\mathbb{R}^n)$ in the setting of Herz spaces. In this section, we will introduce weak Herz spaces.

For $k \in \mathbb{Z}$, set $m_k(\sigma,\, f) = |\{x \in A_k : |f(x)| > \sigma\}|$. Let $\widetilde{m}_k(\sigma,\, f) = m_k(\sigma,\, f)$ for $k \in \mathbb{N}$ and $\widetilde{m}_0(\sigma,\, f) = |\{x \in B(0,\, 1) : |f(x)| > \sigma\}|$.

Definition 1.4.1 Let $0 \in \mathbb{R}$, $0 < q < \infty$ and $0 < p \leqslant \infty$.

(i) A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the homogeneous weak Herz spaces $W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ if

$$\|f\|_{W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \Big(\sum_{k\in\mathbb{Z}} 2^{k\alpha p} m_k(\lambda,\, f)^{p/q} \Big)^{1/p} < \infty,$$

where the usual modification is made when $p = \infty$.

(ii) A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the non-homogeneous weak Herz spaces $WK_q^{\alpha,p}(\mathbb{R}^n)$ if

$$\|f\|_{WK_q^{\alpha,\,p}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \Big(\sum_{k=0}^{\infty} 2^{k\alpha p} \widetilde{m}_k(\lambda,\, f)^{p/q} \Big)^{1/p} < \infty,$$

where the usual modification is made when $p = \infty$.

It is obvious that $W\dot{K}_p^{0,\,p}(\mathbb{R}^n) = WK_p^{0,\,p}(\mathbb{R}^n) = L^{p,\,\infty}(\mathbb{R}^n)$.

We know that the Lorentz space $L^{q,\,r}(\mathbb{R}^n)$ are very important in harmonic analysis and the $L^{q,\,\infty}(\mathbb{R}^n)$ is just a special case of Lorentz space (see [Hun] and [SW] for details). It is a natural problem whether there exists a similar theory in the setting of Herz space. To solve this problem, it is natural to introduce Herz-type Lorentz space as follows.

Let $\alpha \in \mathbb{R}$, $0 < q < \infty$ and $0 < p \leqslant \infty$. Define

$$f^*_{\alpha,\,p,\,q}(t) = \inf\Big\{\sigma \in \mathbb{R}^+ : \Big(\sum_{k\in\mathbb{Z}} 2^{k\alpha p} m_k(\sigma,\, f)^{p/q}\Big)^{1/p} \leqslant t\Big\}$$

and

$$\tilde{f}^*_{\alpha,\,p,\,q}(t) = \inf\Big\{\sigma \in \mathbb{R}^+ : \Big(\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\sigma,\, f)^{p/q}\Big)^{1/p} \leqslant t\Big\}$$

with the usual modifications when $p = \infty$.

Definition 1.4.2 Let $\alpha \in \mathbb{R}$, $0 < q < \infty$ and $0 < p \leqslant \infty$.

(i) A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the homogeneous Herz-type Lorentz spaces $L\dot{K}^{\alpha,\,p,\,r}_q(\mathbb{R}^n)$ if

$$\|f\|_{L\dot{K}^{\alpha,\,p,\,r}_q(\mathbb{R}^n)} = \Big\{\int_0^{\infty} \big(t f^*_{\alpha,\,p,\,q}(t)\big)^r \frac{\mathrm{d}t}{t}\Big\}^{1/r} < \infty$$

with $0 < r < \infty$, and

$$\|f\|_{L\dot{K}^{\alpha,\,p,\,\infty}_q(\mathbb{R}^n)} = \sup_{t>0} \big\{t f^*_{\alpha,\,p,\,q}(t)\big\} < \infty.$$

(ii) A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the non-homogeneous Herz-type Lorentz spaces $LK^{\alpha,\,p,\,r}_q(\mathbb{R}^n)$ if

$$\|f\|_{LK^{\alpha,\,p,\,r}_q(\mathbb{R}^n)} = \Big\{\int_0^{\infty} \big(t \tilde{f}^*_{\alpha,\,p,\,q}(t)\big)^r \frac{\mathrm{d}t}{t}\Big\}^{1/r} < \infty$$

with $0 < r < \infty$, and

$$\|f\|_{LK^{\alpha,\,p,\,\infty}_q(\mathbb{R}^n)} = \sup_{t>0} \big\{t \tilde{f}^*_{\alpha,\,p,\,q}(t)\big\} < \infty.$$

We see that $L\dot{K}^{0,\,q,\,r}_q(\mathbb{R}^n) = LK^{0,\,q,\,r}_q(\mathbb{R}^n) = L^{q,\,r}(\mathbb{R}^n)$, the standard Lorentz spaces. In particular, $L\dot{K}^{0,\,q,\,q}_q(\mathbb{R}^n) = LK^{0,\,q,\,q}_q(\mathbb{R}^n) = L^q(\mathbb{R}^n)$.

To give out the close relationship of the weak type Herz spaces and the Herz-type Lorentz spaces, we first establish a lemma.

Lemma 1.4.1 *Suppose $\{f^l\}$ is a sequence of measurable functions such that, for all $x \in \mathbb{R}^n$, $|f^l(x)| \leqslant |f^{l+1}(x)|$, $l = 1, 2, 3, \cdots$. If f is a measurable function satisfying $|f(x)| = \lim_{l\to\infty} |f^l(x)|$ for $x \in \mathbb{R}^n$, then*

(i) *for each $\sigma > 0$ and $k \in \mathbb{Z}$, $m_k(\sigma, f^l)$ ($\widetilde{m_k}(\sigma, f^l)$ resp.) increases monotonically to $m_k(\sigma, f)$ ($\widetilde{m_k}(\sigma, f)$ resp.) as m tends to ∞.*

(ii) *for each $t > 0$, $\alpha \in \mathbb{R}$, $0 < p \leqslant \infty$ and $0 < q < \infty$, $\left(f^l_{\alpha, p, q}\right)^*(t)$ ($\left(\widetilde{f^l}_{\alpha, p, q}\right)^*(t)$ resp.) increases monotonically to $f^*_{\alpha, p, q}(t)$ ($\tilde{f}^*_{\alpha, p, q}(t)$, resp.).*

Proof Note that

$$\begin{aligned} E^l_{k, \sigma} &= \{x \in A_k : |f^l(x)| > \sigma\} \subset E_{k, \sigma} \\ &= \{x \in A_k : |f(x)| > \sigma\} \text{ and } \cup_{m=1}^{\infty} E^l_\sigma = E^l_\sigma. \end{aligned}$$

It is easy to verify that $m_k(\sigma, f^l) \leqslant m_k(\sigma, f^{l+1})$, $m_k(\sigma, f^l) = |E^l_{k,\sigma}| \leqslant |E_{k, \sigma}| = m_k(f, \sigma)$ and $\lim_{l\to\infty} m_k(\sigma, f^l) = m_k(\sigma, f)$. This proves (i). For each fixed $t > 0$, by the definition of $f^*_{\alpha, p, q}$, we see that $\left(f^l_{\alpha, p, q}\right)^*(t) \leqslant \left(f^{l+1}_{\alpha, p, q}\right)^*(t)$ for all $l = 1, 2, \cdots$. Let $A = \lim_{l\to\infty} \left(f^l_{\alpha, p, q}\right)^*(t)$. It is obvious that $A \leqslant f^*_{\alpha, p, q}(t)$. On the other hand, we have $A \geqslant \left(f^l_{\alpha, p, q}\right)^*(t)$ and for each $\delta > A$,

$$\left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha p} m_k(\delta, f^l)^{p/q}\right\}^{1/p} \leqslant t$$

(for otherwise by the definition of $\left(f^l_{\alpha, p, q}\right)^*(t)$, it follows that $\delta \leqslant \left(f^l_{\alpha, p, q}\right)^*(t)$). Thus,

$$\left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha p} m_k(\delta, f)^{p/q}\right\}^{1/p} \leqslant t.$$

A familiar formula

$$m_k(A, f) = \lim_{\delta > A, \delta\to A} m_k(\delta, f)$$

gives us that

$$\left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha p} m_k(A, f)^{p/q}\right\}^{1/p} \leqslant t.$$

This in turn implies that

$$f^*_{\alpha, p, q}(t) \leqslant A$$

and then gives our desired result. □

The following result shows that weak type Herz spaces and the Herz-type Lorentz spaces are closly related.

Theorem 1.4.1 *Let $\alpha \in \mathbb{R}$, $0 < p,\ q \leqslant \infty$, then $L\dot{K}_q^{\alpha,\,p,\,\infty}(\mathbb{R}^n) = W\dot{K}_q^{\alpha,\,p}(\mathbb{R}^n)$ and $LK_q^{\alpha,\,p,\,\infty}(\mathbb{R}^n) = WK_q^{\alpha,\,p}(\mathbb{R}^n)$.*

Proof We only prove the homogeneous case for $0 < p < \infty$. The arguments for other cases are similar and will be omitted for brevity. By Lemma 1.4.1 and a standard limiting procedure, it suffices to show that $\|f\|_{L\dot{K}_q^{\alpha,\,p,\,\infty}(\mathbb{R}^n)} = \|f\|_{W\dot{K}_q^{\alpha,\,p}(\mathbb{R}^n)}$ for any simple function f. Without loss of generality, we may assume that $f = f_1 + f_3$, $f_1 = C_1^{(1)}\chi_{E_1^{(1)}} + C_2^{(1)}\chi_{E_2^{(1)}}$ and $f_3 = C_1^{(3)}\chi_{E_1^{(3)}} + C_2^{(3)}\chi_{E_2^{(3)}}$, where $E_1^{(i)} \cap E_2^{(i)} = \emptyset$ $(i = 1,\ 3)$, $\operatorname{supp} f_1 \subset A_1$, $\operatorname{supp} f_3 \subset A_3$ and $C_2^{(3)} < C_2^{(1)} < C_1^{(3)} < C_1^{(1)}$. Let $d_1^{(i)} = |E_1^{(i)}|$ and $d_2^{(i)} = |E_1^{(i)}| + |E_2^{(i)}|$. Then

$$m_i(\lambda,\ f) = \begin{cases} 0, & C_1^{(i)} \leqslant \lambda, \\ d_1^{(i)}, & C_2^{(i)} \leqslant \lambda < C_1^{(i)}, \\ d_2^{(i)}, & 0 \leqslant \lambda < C_2^{(i)}, \end{cases}$$

where $i = 1,\ 3$. This in turn implies that

$$\begin{aligned}\|f\|_{W\dot{K}_q^{\alpha,\,p}(\mathbb{R}^n)} &= \sup_{\lambda>0} \lambda\left\{2^{\alpha p} m_1(\lambda,\ f)^{p/q} + 2^{3\alpha p} m_3(\lambda,\ f)^{p/q}\right\}^{1/p} \\ &= \sup\left\{C_2^{(3)}a_1,\ C_2^{(1)}a_2,\ C_1^{(3)}a_3,\ C_1^{(1)}a_4\right\},\end{aligned}$$

where

$$\begin{aligned} a_1 &= \left(2^{\alpha p}(d_2^{(1)})^{p/q} + 2^{3\alpha p}(d_2^{(3)})^{p/q}\right)^{1/p}, \quad a_2 = \left(2^{\alpha p}(d_2^{(1)})^{p/q} + 2^{3\alpha p}(d_1^{(3)})^{p/q}\right)^{1/p}, \\ a_3 &= \left(2^{\alpha p}(d_1^{(1)})^{p/q} + 2^{3\alpha p}(d_1^{(3)})^{p/q}\right)^{1/p}, \quad a_4 = 2^{\alpha}(d_1^{(1)})^{1/q}. \end{aligned}$$

Obviously, $0 < a_4 < a_3 < a_2 < a_1$. A trivial computation then gives that

$$f^*_{\alpha,\,p,\,q}(t) = \begin{cases} C_1^{(1)}, & 0 \leqslant t < a_4, \\ C_1^{(3)}, & a_4 \leqslant t < a_3, \\ C_2^{(1)}, & a_3 \leqslant t < a_2, \\ C_2^{(3)}, & a_2 \leqslant t < a_1, \\ 0, & a_1 \leqslant t. \end{cases}$$

Therefore,

$$\begin{aligned}\|f\|_{L\dot{K}_q^{\alpha,\,p,\,\infty}(\mathbb{R}^n)} &= \sup_{t>0}\{t f^*_{\alpha,\,p,\,q}(t)\} \\ &= \sup\left\{a_4 C_1^{(1)},\ a_3 C_1^{(3)},\ a_2 C_2^{(1)},\ a_1 C_2^{(3)}\right\} = \|f\|_{W\dot{K}_q^{\alpha,\,p}(\mathbb{R}^n)}. \qquad \square\end{aligned}$$

There exists another class of spaces which are also closely related to the Herz-type Lorentz spaces, and these spaces are very important in the interpolation of Herz spaces. Let $f_k^*(t) = \inf\{\sigma \in \mathbb{R}^+ : m_k(\sigma, f) \leqslant t\}$ for $k \in \mathbb{Z}$ and $\tilde{f}_k^*(t) = \inf\{\sigma \in \mathbb{R}^+ : \tilde{m}_k(\sigma, f) \leqslant t\}$ for $k \in \mathbb{N} \cup \{0\}$.

Definition 1.4.3 Let $\alpha \in \mathbb{R}$, $0 < p, q \leqslant \infty$.

(i) A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the spaces $L\dot{K}_q^{\alpha, q}(\mathbb{R}^n)$ if

$$\|f\|_{L\dot{K}_q^{\alpha, p}(\mathbb{R}^n)} = \Big\{ \int_0^\infty \Big(\sum_{k \in \mathbb{Z}} 2^{k\alpha p} \big(t^{1/q} f_k^*(t)\big)^p \Big) \frac{\mathrm{d}t}{t} \Big\}^{1/p} < \infty,$$

where the usual modification is made when $p = \infty$.

(ii) A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the $LK_q^{\alpha, q}(\mathbb{R}^n)$ if

$$\|f\|_{LK_q^{\alpha, p}(\mathbb{R}^n)} = \Big\{ \int_0^\infty \Big(\sum_{k \in \mathbb{Z}} 2^{k\alpha p} \big(t^{1/q} \tilde{f}_k^*(t)\big)^p \Big) \frac{\mathrm{d}t}{t} \Big\}^{1/p} < \infty,$$

where the usual modification is made when $p = \infty$.

For the relationship of the spaces $L\dot{K}_q^{\alpha, p, r}(\mathbb{R}^n)$ and $L\dot{K}_q^{\alpha, p}(\mathbb{R}^n)$, we have the following result.

Theorem 1.4.2 *Let $\alpha \in \mathbb{R}$, $0 < p, q \leqslant \infty$, then $L\dot{K}_q^{\alpha, p}(\mathbb{R}^n) = L\dot{K}_q^{\alpha, p, p}(\mathbb{R}^n)$ and $LK_q^{\alpha, p}(\mathbb{R}^n) = LK_q^{\alpha, p, p}(\mathbb{R}^n)$.*

Proof We only consider the homogeneous case with $0 < p < \infty$. As in the proof of Theorem 1.4.1, it is enough to show that $\|f\|_{L\dot{K}_q^{\alpha, p, p}(\mathbb{R}^n)} \approx \|f\|_{L\dot{K}_q^{\alpha, p}(\mathbb{R}^n)}$ for any simple function. Let $f = f_1 + f_3$ be the same as in the proof Theorem 1.4.1, a straightforward computation leads to that

$$\begin{aligned}
\|f\|_{L\dot{K}_q^{\alpha, p, p}(\mathbb{R}^n)} &= \Big\{ \int_0^\infty \big(t f_{\alpha, p, q}^*(t)\big)^p \frac{\mathrm{d}t}{t} \Big\}^{1/p} \\
&= \Big\{ \int_0^{a_4} t^{p-1} (C_1^{(1)})^p \mathrm{d}t + \int_{a_4}^{a_3} t^{p-1} (C_1^{(3)})^p \mathrm{d}t \\
&\quad + \int_{a_3}^{a_2} t^{p-1} (C_2^{(1)})^p \mathrm{d}t + \int_{a_2}^{a_1} t^{p-1} (C_2^{(3)})^p \mathrm{d}t \Big\}^{1/p} \\
&= p^{-1/p} \big\{ 2^{\alpha p} (C_1^{(1)})^p (d_1^{(1)})^{p/q} + 2^{\alpha p} (C_2^{(1)})^p \big((d_2^{(1)})^{p/q} - (d_1^{(1)})^{p/q} \big) \\
&\quad + 2^{3\alpha p} (C_1^{(3)})^p (d_1^{(3)})^{p/q} + 2^{3\alpha p} (C_2^{(3)})^p \big((d_2^{(3)})^{p/q} - (d_1^{(3)})^{p/q} \big) \big\}^{1/p}.
\end{aligned}$$

On the other hand, it is easy to verify that for $i = 1, 3$,

$$f_i^*(t) = \begin{cases} C_1^{(i)}, & 0 \leqslant t < d_1^{(i)}, \\ C_2^{(i)}, & d_1^{(i)} \leqslant t < d_2^{(i)}, \\ 0, & t \geqslant d_2^{(i)}, \end{cases}$$

Thus,

$$\begin{aligned}\|f\|_{L\dot{K}_q^{\alpha,\,p}(\mathbb{R}^n)} &= \Big\{\int_0^\infty 2^{\alpha\,p}\big(t^{1/q}f_1^*(t)\big)^{1/p}\frac{\mathrm{d}t}{t} + \int_0^\infty 2^{3\alpha\,p}\big(t^{1/q}f_3^*(t)\big)^{1/p}\frac{\mathrm{d}t}{t}\Big\}^{1/p} \\ &= \Big\{\int_0^{d_1^{(1)}} 2^{\alpha p}t^{p/q-1}(C_1^{(1)})^p\mathrm{d}t + \int_{d_1^{(1)}}^{d_2^{(1)}} 2^{\alpha p}t^{p/q-1}(C_2^{(1)})^p\mathrm{d}t \\ &\quad + \int_0^{d_1^{(3)}} 2^{3\alpha p}t^{p/q-1}(C_1^{(3)})^p\mathrm{d}t + \int_{d_1^{(3)}}^{d_2^{(3)}} 2^{3\alpha p}t^{p/q-1}(C_2^{(3)})^p\mathrm{d}t\Big\}^{1/p} \\ &= (qp^{-1})^{1/p}\big\{2^{\alpha p}(C_1^{(1)})^p(d_1^{(1)})^{p/q} + 2^{\alpha p}(C_2^{(1)})^p\big((d_2^{(1)})^{p/q} - (d_1^{(1)})^{p/q}\big) \\ &\quad + 2^{3\alpha p}(C_1^{(3)})^p(d_1^{(3)})^{p/q} + 2^{3\alpha p}(C_2^{(3)})^p\big((d_2^{(3)})^{p/q} - (d_1^{(3)})^{p/q}\big)\big\}^{1/p} \\ &\approx \|f\|_{L\dot{K}_q^{\alpha,\,p,\,p}(\mathbb{R}^n)}.\end{aligned}$$

This is our desired estimate. □

Similar to the proof of Theorem 1.4.1, we can obtain

Theorem 1.4.3 *Let $\alpha \in \mathbb{R}$, $1 < q < \infty$, then $L\dot{K}_q^{\alpha,\,q,\,q}(\mathbb{R}^n) = L\dot{K}_q^{\alpha,\,q}(\mathbb{R}^n) = \dot{K}_q^{\alpha,\,q}$ and $LK_q^{\alpha,\,q,\,q}(\mathbb{R}^n) = LK_q^{\alpha,\,q}(\mathbb{R}^n) = K_q^{\alpha,\,q}(\mathbb{R}^n)$.*

1.5 Notes and related results

- The Beurling algebra $A^q(\mathbb{R}^n)$, introduced by Beurling [Be], coincide with space $K_q^{n(1-1/q),1}(\mathbb{R}^n)$ defined in (b) of Definition 1.1.1, and is equipped with a different, but equivalent norm. The norm of the spaces A_q $(K_q^{n(1-1/q),\,1}(\mathbb{R}^n))$ used in Definition 1.1.1 was first introduced by Feichtinger [Fe]. in Definition 1.1.1. It was Herz [Hr] who first considered the homogeneous Herz space. spaces of type of $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$. To characterize certain properties of functions, Herz introduced a function space K_q, which is known equivlent to $\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$ but with different notations and notations. Flett [Fl] gave a characterization of the Herz spaces which is easily seen to be equivalent to Definition 1.1.1.

- Lu and Yang [LY12] introduced the Herz-type Sobolev and considered their relation with Herz spaces, from which Lu and Yang established some regularity of linear quantities appearing in the compensated compactness theory on Herz-type spaces. The result of Lu and Yang are generalization of the result of Coifman, Lions, Meyer and Semmes [CLMS].

- The content of Section 1.2 is taken from the paper of Henandez and Yang [HY2]. The results of Theorem 1.2.1 for the spaces $\dot{\ell}_p^0(\mathcal{A})$ and $\dot{C}_0^0(\mathcal{A})$ with $1 \leqslant p < \infty$ and $A_j = A$ for $j \in \mathbb{Z}$ can be found in [Tr, p. 68].

- If $1 < q < \infty$, the space $(K_q^{-n/q,\infty})_0(\mathbb{R}^n)$ was introduced by Feichtinger [Fe] and García-Cuerva [Ga]. For spaces $K_q^{n(1-1/q),1}(\mathbb{R}^n)$ and $K_q^{-n/q,\infty}(\mathbb{R}^n)$, Corollary 1.2.1 has been obtained by Feichtinger in [Fe, Theorem 3], and García-Cuerva [Ga, Theorem 1.4]. For spaces $\dot{K}_q^{n(1/p-1/q),p}(\mathbb{R}^n)$ and $\dot{K}_q^{-n/q,\infty}(\mathbb{R}^n)$, Corollary 1.2.1 has been obtained by García-Cuerva and Herrero in [GaH, Proposition 1.7]. For the space $K_q^{n(1-1/q),1}(\mathbb{R}^n)$, Corollary 1.2.2 has been obtained by García-Cuerva (see Corollary 1.5 in [Ga]).

- The notation "block" was first introduced by Taibleson and Weiss in [TW1], in order to study the almost everywhere convergence of the Fourier series, see also [LTaW]. For the case of $\alpha = n(1-1/q)$, a central block is just a block in Taibleson- Weiss sense at the origin. The content of this section is due to Lu and Yang [LY5].

- The content of Section 1.4 is from the work of Hu, Lu and Yang [HLY1]. It should be pointed out that the space $L\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $LK_q^{\alpha,p}(\mathbb{R}^n)$ were introduced by Yang and Hernández in [HY3]

Chapter 2

Herz-type Hardy Spaces

During 1970$\sim$1980, there has been many significant progress in the study of Hardy spaces. Some prototypical works in this area are the paper [FS], [TW2] and [La] etc., see also [Lu1]. Thanks to the real-variable characterization, the atomic decomposition theory and the molecule decomposition theory, the Hardy space is now very important in harmonic analysis, and plays a very important role in many related fields, such as PDE and complex analysis.

The study of the theory of Herz type Hardy spaces, which originated in the late of 1980's in order to give a natural analog to the classical Hardy spaces, has now been a very interesting topics in Harmonic analysis. As we will see in this chapter, the Herz type Hardy space is a suitable local version of the Hardy spaces, and enjoys some interesting characterizations.

2.1 Definitions and basic properties

The definition of the Herz-type Hardy space is in a way similar to that of the classical $H^p(\mathbb{R}^n)$ spaces, in terms of the grand maximal operator. For $f \in \mathscr{S}'(\mathbb{R}^n)$, the grand maximal function of f is defined by

$$G_N f(x) = \sup_{\phi \in \mathscr{A}_N} \left|\phi^*_\nabla(f)(x)\right|, \tag{2.1.1}$$

where $N > n+1$, $\mathscr{A}_N = \left\{\phi \in \mathscr{S}(\mathbb{R}^n) : \sup_{|\alpha|,\,|\beta| \leqslant N} |x^\alpha D^\beta \phi(x)| \leqslant 1\right\}$, ϕ^*_∇ is the nontangent maximal operator defined by

$$\phi^*_\nabla(f)(x) = \sup_{|y-x|<t} |\phi_t * f(y)| \tag{2.1.2}$$

with $\phi_t(x) = t^{-n}\phi(x/t)$.

Definition 2.1.1 Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $1 < q < \infty$ and $N > n+1$.

(i) The homogeneous Herz-type Hardy space $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \left\{f \in \mathscr{S}'(\mathbb{R}^n) : G_N f \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)\right\}$$

and we define $\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \|G_N f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$.

(ii) The non-homogeneous Herz-type Hardy space $HK_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$HK_q^{\alpha,p}(\mathbb{R}^n) = \left\{f \in \mathscr{S}'(\mathbb{R}^n) : G_N f \in K_q^{\alpha,p}(\mathbb{R}^n)\right\}$$

and we define $\|f\|_{HK_q^{\alpha,p}(\mathbb{R}^n)} = \|G_N f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}$.

Definition 2.1.2 Let $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $1 < q < \infty$, $N > n+1$.

(i) The homogeneous weak Herz-type Hardy space $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$WH\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \left\{f \in \mathscr{S}'(\mathbb{R}^n) : G_N f \in W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)\right\}$$

and we define $\|f\|_{WH\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \|G_N f\|_{W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$.

(ii) The non-homogeneous weak Herz-type Hardy space $WHK_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$WHK_q^{\alpha,p}(\mathbb{R}^n) = \left\{f \in \mathscr{S}'(\mathbb{R}^n) : G_N f \in WK_q^{\alpha,p}(\mathbb{R}^n)\right\}$$

and we define $\|f\|_{WHK_q^{\alpha,p}(\mathbb{R}^n)} = \|G_N f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}$.

We will see from the proof Theorem 2.1.1, the definitions of $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $HK_q^{\alpha,p}(\mathbb{R}^n)$ do not depend on the choice of the number N provided $N > n+1$.

The following theorem gives out the relation of $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ $(HK_q^{\alpha,p}(\mathbb{R}^n))$ and $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ $(K_q^{\alpha,p}(\mathbb{R}^n))$.

Proposition 2.1.1 *Let $0 < p < \infty$, $1 < q < \infty$.*

(1) *If $-n/q < \alpha < n(1-1/q)$, then*

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) \cap L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$$

and

$$HK_q^{\alpha,p}(\mathbb{R}^n) \cap L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) = K_q^{\alpha,p}(\mathbb{R}^n).$$

(2) *If $n(1-1/q) \leqslant \alpha < \infty$, then*

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) \cap L^q_{\mathrm{loc}}(\mathbb{R}^n) \subsetneqq \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$$

and

$$HK_q^{\alpha,p}(\mathbb{R}^n) \cap L^q_{\mathrm{loc}}(\mathbb{R}^n) \subsetneqq K_q^{\alpha,p}(\mathbb{R}^n).$$

Proof We only consider the homogeneous spaces. For (1), if $f \in H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)\cap L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$, it is well known that for some positive constant C,

$$|f(x)| \leqslant CG_N(f)(x)$$

and so $f \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$. On the other hand, by the boundedness result on Herz spaces for sublinear operator, we can prove that the grand maximal operator G_N is bounded on $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ provided that $1 < q < \infty$ and $-n/q < \alpha < n(1-1/q)$.[1] This along with Proposition 1.1.3 gives us conclusion (1).

To show the conclusion (2), note that the inclusion

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) \cap L^q_{\mathrm{loc}}(\mathbb{R}^n) \subset \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$$

is still true as in (1). On the other hand, let $n(1-1/q) \leqslant \alpha < \infty$ and $f \in H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, the central (α, q)-atom decomposition[2] implies that f has integral zero, and so

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) \cap L^q_{\mathrm{loc}}(\mathbb{R}^n) \neq \dot{K}_q^{\alpha,p}(\mathbb{R}^n).$$

□

By Proposition 1.1.2 and the $L^q(\mathbb{R}^n)$-boundedness $(1 < q < \infty)$ of the grand maximal operator G_N, it is easy to deduce the following conclusion.

Proposition 2.1.2 *Let $\alpha \in \mathbb{R}$, $p \in (0, \infty)$ and $q \in (1, \infty)$. Then*

$$HK_q^{\alpha,p}(\mathbb{R}^n) = L^q(\mathbb{R}^n) \cap H\dot{K}_q^{\alpha,p}(\mathbb{R}^n).$$

Proposition 2.1.1 tells us that for the Herz-type Hardy space $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, the interesting case is that $\alpha \geqslant n(1-1/q)$ (this, in some sense, explain why we only give the "building block decomposition" of $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ for the case that $\alpha \geqslant n(1-1/q)$ in next two sections).

Except the grand maximal operator G_N, the space $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ ($HK_q^{\alpha,p}(\mathbb{R}^n)$) enjoys some other characterizations in terms of some other maximal operators. To give these characterizations, we first introduce some maximal operator.

Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ with integral 1. For $t > 0$, set $\phi_t(x) = t^{-n}\phi(x/t)$. For $f \in \mathscr{S}'(\mathbb{R}^n)$, define the maximal operator ϕ^*_+ by

$$\phi^*_+(f)(x) = \sup_{t>0} |f * \phi_t(x)|.$$

1 We will prove this in Section 5.1.
2 See Theorem 2.2.1 in Section 2.2.