p-adic Representations, θ-Correspondence and the Langlands-Shahidi Theory

（局部群表示论，θ-对应和Langlands-Shahidi方法）

Edited by Ye Yangbo(叶扬波) & Tian Ye(田野)

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Preface

This book is the first volume of the [illegible], which is devoted to publishing peer-reviewed workshop lecture notes and the proceedings of conferences on all branches of contemporary number theory research. The series intends to target number theory researchers and students, including both experts and non-experts of the covered subjects.

This book includes notes on five lectures given at p [illegible], a workshop held from June 12 to 21, 2011 at the Morningside Center of Mathematics, Beijing, China that was organized by Tian Ye of Morningside Center of Mathematics and Ye Yangbo of the University of Iowa. The speakers at the workshop include:

Corinne Blondel of Institut de Mathématiques de Jussieu;

Colin Bushnell of King's College;

Daniel File of the University of Iowa;

Muthukrishinan Krishinamurthy of the University of Iowa;

David Manderscheid of Ohio State University;

Vincent Sécherre of l'Université de Versailles Saint-Quentin, and

Freydoon Shahidi of Purdue University.

As suggested by its title, this volume contains lectures on representation theory of p-adic groups. For example, Colin J. Bushnell's lecture notes attempt to derive a complete list of the irreducible cuspidal complex representations of the group $\mathrm{GL}_n(F)$, where F is a non-Archimedean local field. Here, complex representations refer to representations of $\mathrm{GL}_n(F)$ on complex vector spaces. The last part of his notes explores the relationship of complex representations with the local Langlands correspondence.

Corinne Blondel's notes have a more general setting. The groups are assumed to be reductive p-adic groups, and the representations are taken on vector spaces over a commutative field R. After an introduction of this general setup, the characteristic of R is assumed to be zero or positive but not equal to p. The notes prove a major theorem: that smooth irreducible representations of a reductive p-adic group over a field of characteristic not equal to p are admissible.

In the notes by Vincent Sécherre, the group is again the GL_n over a non-Archimedean locally compact field F with residue characteristic p. The representations are taken on vector spaces over an algebraically closed field R. This includes the case of R being the algebraic closure of a finite field of characteristic ℓ. The notes pay special attention to whether ℓ is or is not equal to p. The main theorem proved in the notes is that the category of all smooth representations of $\mathrm{GL}_n(F)$ on R-vector spaces decomposes into a product of indecomposable summands.

The theory of theta correspondence is globally a general construction of automorphic forms of groups over algebraic number fields, and locally a general correspondence between admissible representations of two groups of a reductive dual pair. The theory has its origin from classical theta series which are modular forms. David Manderscheid's lecture notes provide an introduction to reductive dual pairs and local theta correspondence.

The Langlands-Shahidi method is a powerful method that uses the Eisenstein series to obtain functional equations and analytic properties of certain automorphic L-functions. These L-functions are those appear in constant terms and other Fourier coefficients of the Eisenstein series. The notes by Freydoon Shahidi prove the main results of Langlands on analytic properties of the Eisenstein series. These notes provide the part of the theory of the Eisenstein series that is needed in the author's [illegible] L [illegible] to develop the theory of automorphic L-functions.

The editors of this volume would like to express their sincere thanks to the authors for their contributions, and to the referees for their valuable comments and suggestions. Heartfelt gratitude is due to the Morningside Center of Mathematics for their grant, organization, and facility support. We are also greatly indebted to Jing Ma of Jilin University who compiled the index for this volume and to Zhao Yanchao of Science Press for the careful preparation of this book for publication.

Ye Yangbo
Iowa City, Iowa
March, 2013

Contents

1. Arithmetic of Cuspidal Representations

Colin J. Bushnell*

We aim to write down a complete list of the irreducible cuspidal (complex) representations of the group $\mathrm{GL}_n(F)$, where F is a non-Archimedean local field. We indicate how this list relates to the local Langlands correspondence.

The text is a lightly edited and expanded version of the notes distributed at the Summer School and its tone remains informal. It relies for background on some of the other courses. In the earlier parts, the useful forms of several results emerged gradually and are not so easy to track down in the literature. We have therefore paid them a bit more attention but, otherwise, proofs are rather rare. The core of the subject is fully exposed in [5] so, beyond the first half of the notes, we have given only an introduction to the definitions and a high level overview of the argument. We have omitted entirely one of the main themes of [5], concerned with non-vanishing of Jacquet modules: while crucial to the arguments, it does not enter visibly into the description of the cuspidal spectrum on which we focus here. The last part of the notes, concerning the relation with the Langlands correspondence, represents a work in progress and relies on a substantial literature. The area is covered more carefully, with fuller references, in a parallel set of notes [2] which the interested reader might find helpful.

As indicated, we give little by way of detailed references, only some "further reading". Beyond [5], many of the core concepts originate in two very old papers [1], [6]. These remain accessible and helpful to readers new to the area.

1.1 Cuspidal representations by induction

As a starting point, we review a standard method of producing irreducible cuspidal representations. Much of the background we use here may be found in other papers in this volume. The reader may also find helpful the early pages of [4].

* The author wishes to record his deep appreciation of the organizers, their supporters and students for making the Summer School a stimulating and supremely enjoyable experience.

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1.1.1 Background and notation

Throughout, F is a non-Archimedean local field, with discrete valuation ring $\mathfrak{o}_F$ and residue field $\Bbbk_F = \mathfrak{o}_F/\mathfrak{p}_F$. We set $U_F = U_F^0 = \mathfrak{o}_F^\times$ and $U_F^m = 1+\mathfrak{p}_F^m$, $m \geqslant 1$.

We usually work with the general linear group $G = \mathrm{GL}_n(F)$, and the full matrix algebra $A = M_n(F)$. Sometimes, it is better to think of G as $\mathrm{Aut}_F(W)$, where W is an F-vector space of dimension n, and then $A = \mathrm{End}_F(W)$.

If (π, V) is a smooth representation of G, then $(\check\pi, \check V)$ denotes the [illegible] of (π, V). A [illegible] π is a function $G \to \mathbb{C}$ that is a finite linear combination of functions $g \mapsto \langle \check v, \pi(g)v\rangle$, for $v \in V$ and $\check v \in \check V$.

If K is a compact open subgroup of G, then V^K is the space of $v \in V$ such that $\pi(k)v = v$, $k \in K$.

Suppose that (π, V) is [illegible] Thus (π, V) is [illegible]: $\dim V^K < \infty$, for all K. Moreover, $(\check\pi, \check V)$ is irreducible and admissible. The second dual $(\check\pi, \check V)^\vee$ is isomorphic to (π, V). One says that (π, V) is [illegible] if all coefficients of π are compactly supported modulo the centre $F^\times$ of G. Indeed, if one non-zero coefficient has this property, so do all.

We return to a general smooth representation (π, V) of G. Let P be a parabolic subgroup of G. Thus, thinking of G as $\mathrm{Aut}_F(W)$, P is the stabilizer of a flag of subspaces

$$W = W_0 \supset W_1 \supset \cdots \supset W_r = \{0\}.$$

The unipotent radical N of P is the subgroup of P consisting of elements acting trivially on all quotients W_i/W_{i+1}, $0 \leqslant i < r$. The [illegible] V_N of (π, V) at N is the maximal quotient of V on which N acts trivially: it is the quotient of V by the subspace spanned by all $v-\pi(n)v$, $v \in V$, $n \in N$.

Key fact. [illegible] (π, V) [illegible] G [illegible]:

(1) [illegible] (π, V) [illegible]

(2) [illegible] P [illegible] G [illegible] V_N [illegible] N [illegible] P [illegible]

1.1.2 Intertwining and Hecke algebras

Let K be a compact open subgroup of G, and (ρ, W) a smooth representation of K. Thus ρ is [illegible]: it is a direct sum of irreducible representations, the irreducible factors being unique up to equivalence and permutation.

If $g \in G$, write $K^g = g^{-1}Kg$ and let ρ^g be the representation $x \mapsto \rho(gxg^{-1})$ of K^g.

For $i = 1, 2$, let K_i be a compact open subgroup of G and (ρ_i, W_i) a smooth representation of K_i. Let $g \in G$. We say g [illegible] ρ_1 [illegible] ρ_2 if

$$\mathrm{Hom}_{K_1^g \cap K_2}(\rho_1^g, \rho_2) \neq 0.$$

In other words, the (semisimple) representations $\rho_1^g|_{K_1^g \cap K_2}$, $\rho_2|_{K_1^g \cap K_2}$ have an irreducible component in common. This property depends only on the double coset $K_1 g K_2$. Note that, if g intertwines ρ_1 with ρ_2, then g^{-1} intertwines ρ_2 with ρ_1.

Remark. Suppose that the representations ρ_i above are both irreducible. Let (π, V) be an irreducible smooth representation of G. Say π [illegible] ρ_i if $\mathrm{Hom}_{K_i}(\rho_i, \pi) \neq 0$. If there exists an irreducible representation π containing both ρ_i, then ρ_1 intertwines with ρ_2 in G.

We take a compact open subgroup K of G, and an irreducible smooth representation (ρ, W) of G. Let $\mathcal{H}(G, \rho)$ be the space of compactly supported functions

$$\phi : G \longrightarrow \mathrm{End}_{\mathbb{C}}(W),$$

$$\phi(k_1 g k_2) = \rho(k_1) \circ \phi(g) \circ \rho(k_2),$$

for $g \in G$ and $k_1, k_2 \in K$.

Lemma. [illegible] $g \in G$ [illegible]:
(1) g [illegible] ρ
(2) [illegible] $\phi \in \mathcal{H}(G, \rho)$ [illegible] $\phi(g) \neq 0$
[illegible] $g \in G$ [illegible] $\mathcal{H}(G, \rho)$ [illegible] KgK [illegible] $\mathrm{Hom}_{K^g \cap K}(\rho^g, \rho)$

The space $\mathcal{H}(G, \rho)$ is a $\mathbb{C}$-algebra under convolution relative to a Haar measure μ_G on G. The function $\boldsymbol{e}_\rho$, with support K and such that

$$\boldsymbol{e}_\rho(k) = \mu_G(K)^{-1} \rho(k), \quad k \in K,$$

provides the unit element of $\mathcal{H}(G, \rho)$. The algebra $\mathcal{H}(G, \rho)$ is the ρ [illegible] G

Comment. In the literature, our $\mathcal{H}(G, \rho)$ is sometimes viewed as $\mathcal{H}(G, \check{\rho})$. Each viewpoint has its own advantages and disadvantages. Whichever choice one makes, $\mathcal{H}(G, \check{\rho})$ is linearly anti-isomorphic to $\mathcal{H}(G, \rho)$.

1.1.3 Compact induction

Let (ρ, W) be a smooth representation of the compact open subgroup K of G. We form the compactly induced representation $c\text{-}\mathrm{Ind}_K^G \rho$ in the usual manner. The vector space underlying this representation consists of all compactly supported functions $f : G \to W$ such that

$$f(kg) = \rho(k)f(g), \quad g \in G,\ k \in K,$$

and the group G acts by right translation. It is necessary to remember that we have a canonical map $W \to c\text{-Ind}\rho$: a vector $w \in W$ gets mapped to the function ϕ_w^0 with support K such that $\phi_w^0(1_G) = w$.

Suppose that (ρ, W) is irreducible. Choose a Haar measure μ_G on G. For $f \in c\text{-Ind}\rho$ and $\phi \in \mathcal{H}(G,\rho)$, the function

$$\phi * f(g) = \int_G \phi(x) f(x^{-1}g)\, d\mu_G(x),$$

again lies in c-Indρ. This gives us an algebra homomorphism

$$\mathcal{H}(G,\rho) \longrightarrow \mathrm{End}_G(c\text{-Ind}\rho). \tag{1.1}$$

Proposition. [illegible] (1.1) [illegible]

Let (π, V) be a smooth representation of G. Frobenius Reciprocity gives

$$\mathrm{Hom}_K(\rho,\pi) = \mathrm{Hom}_G(c\text{-Ind}\rho, \pi).$$

The space $V_\rho = \mathrm{Hom}_K(\rho,\pi)$ thus becomes a right $\mathcal{H}(G,\rho)$-module: if $\Phi : c\text{-Ind}\rho \to \pi$ is a G-homomorphism and $\phi \in \mathcal{H}(G,\rho)$, then $\Phi\phi$ is the map sending f to $\Phi(\phi * f)$, $f \in c\text{-Ind}\rho$. The following is good to know, but we will not use it openly.

Fact. [illegible] (π, V) [illegible] ρ [illegible] $\mathcal{H}(G,\rho)$ [illegible] V_ρ [illegible] $(\pi, V) \mapsto V_\rho$ [illegible] G [illegible] ρ [illegible] $\mathcal{H}(G,\rho)$ [illegible]

We use this discussion in a slightly different context. We fix a smooth character ω of the centre $F^\times$ of G. We consider open subgroups K of G, containing $F^\times$ and such that $K/F^\times$ is compact. Nothing then changes provided we consider only representations (ρ, W) of K such that $\rho(x)w = \omega(x)w$, $w \in W$, $x \in F^\times$, and representations of G with the same property.

Theorem. [illegible] K [illegible] G [illegible] $F^\times$ [illegible] (ρ, W) [illegible] K [illegible] $\mathcal{H}(G,\rho) \cong \mathbb{C}$ [illegible] $\mathbb{C}$ [illegible] $c\text{-Ind}_K^G\rho$ [illegible] G [illegible]

Comment. If c-Indρ is irreducible, then surely $\mathcal{H}(G,\rho) \cong \mathrm{End}_G(c\text{-Ind}\rho)$ must reduce to $\mathbb{C}$ (Schur's Lemma). The hypothesis amounts to the following: $g \in G$ intertwines ρ if and only if $g \in K$.

[illegible] Let us write $c\text{-Ind}\rho = (\pi, V)$. There is a unique character ω of $F^\times$ such that $\rho(z)w = \omega(z)w$, for $z \in F^\times$ and $w \in W$. We then have $\pi(z)v = \omega(z)v$, for $v \in V$.

In particular, V is semisimple as a representation of K. Let V^ρ be the ρ-isotypic component of V: this is the sum of all irreducible K-subspaces equivalent to ρ. So,

$$\mathrm{Hom}_K(W, V^\rho) = \mathrm{Hom}_K(W, V) \cong \mathrm{Hom}_G(V, V) = \mathrm{End}_G(V) \cong \mathcal{H}(G, \rho).$$

However, $\dim \mathcal{H}(G, \rho) = 1$, so $V^\rho \cong W$. We may therefore identify V^ρ: it is the canonical image of W in V. It consists of all functions $v \in V$ with support K. Let U be a non-zero G-subspace of V. Therefore

$$0 \neq \mathrm{Hom}_G(U, V) \subset \mathrm{Hom}_G(U, \mathrm{Ind}_K^G \rho) \cong \mathrm{Hom}_K(U, W).$$

This says $U^\rho = U \cap V^\rho \neq 0$. Since V^ρ is irreducible as K-space, $U^\rho = V^\rho$. However, V^ρ generates V as G-space, so $U = V$.

The G-space $V = c\text{-Ind}\rho$ is irreducible, and so it is admissible. The contragredient $(\check{\pi}, \check{V}) \cong \mathrm{Ind}_K^G \check{\rho}$ is therefore irreducible. We choose $w \in W$, and let $v \in V$ be the function with support K such that $v(1) = w$. We choose $\check{w} \in \check{W}$ such that $\langle \check{w}, w \rangle \neq 0$. We define $\check{v} \in \mathrm{Ind}_K^G \check{\rho}$ in the same way: its support is K and $\check{v}(1) = \check{w}$. For $g \in G$, we have

$$\langle \check{v}, \pi(g) v \rangle = \int_{G/F^\times} \langle \check{v}(x), v(xg) \rangle \, d\dot{\mu}(x).$$

The integrand vanishes unless $x \in K$. For $x \in K$, it vanishes unless $g \in K$. The support of this coefficient is therefore contained in K. For $g = 1$, it reduces to $\dot{\mu}(K/F^\times) \langle \check{w}, w \rangle \neq 0$.

We conclude that $c\text{-Ind}\rho$ has one non-zero coefficient which is compactly supported modulo $F^\times$. Since $c\text{-Ind}\rho$ is irreducible, all coefficients have this property, and so $c\text{-Ind}\rho$ is cuspidal. □

Remark. In this situation, $\mathrm{Hom}_K(\rho, c\text{-Ind}\rho) \cong \mathrm{End}_G(c\text{-Ind}\rho) \cong \mathbb{C}$. Therefore ρ occurs in $c\text{-Ind}\rho$ [illegible]

1.1.4 An example

Let $K_0 = \mathrm{GL}_n(\mathfrak{o}_F)$: this is a maximal compact subgroup of $G = \mathrm{GL}_n(F)$. Let $K = F^\times K_0$ (which is the G-normalizer of K_0). We also need the normal subgroup

$$K_1 = 1 + \mathfrak{p}_F M_n(\mathfrak{o}_F)$$

of K_0. In particular, $\mathcal{G} = K_0/K_1 \cong \mathrm{GL}_n(\Bbbk_F)$. Let $\tilde{\lambda}$ be an irreducible [illegible] representation of $\mathcal{G}$: this means that $\tilde{\lambda}$ does not contain the trivial character of $\mathcal{N}$, for the unipotent radical $\mathcal{N}$ of a proper parabolic subgroup of $\mathcal{G}$. Let λ be the inflation of $\tilde{\lambda}$ to an irreducible representation of K_0. Let $g \in G$ intertwine λ. Only the double coset K_0gK_0 matters, so we may put g in the following canonical form. We choose a prime element ϖ of F, and then g is the diagonal matrix with entries $\varpi^{a_1}, \varpi^{a_2}, \cdots, \varpi^{a_n}$, where the a_i are integers such that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$. If all the a_i are equal, then $g \in K = F^\times K_0$. We therefore assume the contrary: there exists i, $1 \leqslant i \leqslant n-1$, such that $a_i < a_{i+1}$. The group $K_0^g \cap K_0$ therefore contains the group of block matrices

$$N = \begin{pmatrix} I_i & 0 \\ \mathfrak{o}_F & I_{n-i} \end{pmatrix}.$$

The representation λ^g is trivial on this group, but λ does not contain its trivial character. Thus g cannot intertwine λ.

Let Λ be any representation of K such that $\Lambda|_{K_0} \cong \lambda$. Such a representation surely exists. Any $g \in G$ which intertwines Λ must also intertwine λ, and so lie in K. In other words, $c\text{-Ind}_K^G \Lambda$ is an irreducible, cuspidal representation of G.

Looking more carefully at this argument, we get the following.

Proposition. [illegible] λ_1 λ_2 [illegible] K_0 [illegible] K_1 [illegible] λ_1 [illegible] fl [illegible] K_0/K_1 [illegible] λ_1 λ_2 [illegible] G [illegible] $\lambda_2 \cong \lambda_1$

This tells us a little more. Take Λ as before, so that $\lambda = \Lambda|_{K_0}$ is inflated from cuspidal. The group K stabilizes the space of K_1-fixed points in $c\text{-Ind}\Lambda$, which therefore carries a natural representation Λ_1 of K/K_1. The intertwining property tells us that any irreducible component of Λ_1 is equivalent to Λ. Frobenius reciprocity says that Λ occurs in $c\text{-Ind}\Lambda$ with multiplicity one, so Λ_1 is irreducible and equivalent to Λ.

1.1.5 A broader context

Let (π, V) be an irreducible smooth representation of G, and suppose that $V^{K_1} \neq 0$. This space of K_1-fixed vectors is stable under $\pi(K)$, and so carries a representation ρ of K. This is smooth, of finite dimension. Consider an irreducible component Λ of ρ. If $\Lambda|_{K_0}$ is inflated from cuspidal, then $\pi \cong c\text{-Ind}_K^G \Lambda$ and $\Lambda = \rho$.

The only other possibility is that $\lambda = \Lambda|_{K_0}$ is not inflated from cuspidal, for any irreducible component Λ of ρ. A key step in the general development is that, in this case, π has a non-trivial Jacquet module and so cannot be cuspidal. This argument takes some time and space, and we will not enter into it. However, once accepted, we have shown:

Corollary. [illegible] (π, V) [illegible] G [illegible] $V^{K_1} \neq 0$ [illegible] Λ [illegible] K [illegible] K_1 [illegible] $\Lambda|_{K_0}$ [illegible] fl [illegible] $\pi \cong c\text{-Ind}\Lambda$

1.2 Lattices, orders and strata

We introduce some general apparatus, giving us a good supply of compact open subgroups of $G = \mathrm{GL}_n(F)$ and a method of writing down, at least partially, a lot of representations with which we can work. This leads us to another family of examples of cuspidal representations, constructed by induction.

1.2.1 Lattices and orders

Let V be an F-vector space of finite dimension n. We set $A = \mathrm{End}_F(V) \cong M_n(F)$ and $G = \mathrm{Aut}_F(V) \cong \mathrm{GL}_n(F)$. An $\mathfrak{o}_F$ [illegible] V is a finitely generated $\mathfrak{o}_F$-submodule L of V which spans V over F.

Lemma. [illegible] L [illegible] $\mathfrak{o}_F$ [illegible] V [illegible] L [illegible] $\mathfrak{o}_F$ [illegible] V [illegible] F [illegible] $\{v_1, v_2, \cdots, v_n\}$ [illegible] V [illegible] L [illegible] $\mathfrak{o}_F$ [illegible] $\mathfrak{o}_F$ [illegible] V [illegible] n

An $\mathfrak{o}_F$ [illegible] A is an $\mathfrak{o}_F$-lattice $\mathfrak{a}$ in A which is also a subring of A containing 1. If $\mathfrak{a}$ is an $\mathfrak{o}_F$-order in A, an $\mathfrak{a}$-lattice in V is an $\mathfrak{o}_F$-lattice L in V such that $\mathfrak{a}L = L$.

If L is an $\mathfrak{o}_F$-lattice in V, we may form

$$\mathfrak{m}(L) = \{x \in A : xL \subset L\}.$$

If $V = F^n$ and $L = \mathfrak{o}_F^n$, then $\mathfrak{m}(L) = M_n(\mathfrak{o}_F)$.

Proposition. (1) [illegible] L [illegible] $\mathfrak{o}_F$ [illegible] V [illegible] $\mathfrak{m}(L)$ [illegible] $\mathfrak{o}_F$ [illegible] A [illegible] $M_n(\mathfrak{o}_F)$

(2) [illegible] L L' [illegible] $\mathfrak{o}_F$ [illegible] V [illegible] $\mathfrak{m}(L)$ [illegible] G [illegible] $\mathfrak{m}(L')$

(3) [illegible] $\mathfrak{a}$ [illegible] $\mathfrak{o}_F$ [illegible] A [illegible] $\mathfrak{o}_F$ [illegible] L [illegible] V [illegible] $\mathfrak{a} \subset \mathfrak{m}(L)$

[illegible] The first assertion follows from the lemma, as does the second. For the third, let L_0 be some $\mathfrak{o}_F$-lattice in V, and consider the $\mathfrak{a}$-module $L = \mathfrak{a}L_0$ generated by L_0. This is an $\mathfrak{a}$-lattice and $\mathfrak{a} \subset \mathfrak{m}(L)$. □

Orders of the form $\mathfrak{m}(L)$ are called [illegible] for the following reason.

Exercise. Let L be an $\mathfrak{o}_F$-lattice in V and set $\mathfrak{m} = \mathfrak{m}(L)$. Let L' be some $\mathfrak{m}$-lattice in V. Show that $L' = xL$, for some $x \in F^\times$. Deduce that, if $\mathfrak{m}$, $\mathfrak{m}'$ are maximal orders in A, then $\mathfrak{m} \subset \mathfrak{m}'$ if and only if $\mathfrak{m} = \mathfrak{m}'$.

1.2.2 Lattice chains

An $\mathfrak{o}_F$ [illegible] V is a non-empty set $\mathcal{L}$ of $\mathfrak{o}_F$-lattices in V which is linearly ordered under inclusion and such that $xL \in \mathcal{L}$ for any $x \in F^\times$ and $L \in \mathcal{L}$. One may number the elements of $\mathcal{L}$,

$$\mathcal{L} = \{L_i : i \in \mathbb{Z}\},$$

so that $L_i \supsetneq L_{i+1}$, for all i. Stability under scalar multiplication then amounts to:

Lemma. [illegible] $e \geqslant 1$ [illegible] $\varpi L_i = L_{i+e}$ [illegible] $i \in \mathbb{Z}$ [illegible] ϖ [illegible] F

The integer $e = e_F(\mathcal{L})$ is the F-period of $\mathcal{L}$.

We have

$$L_0 \supset L_1 \supset \cdots \supset L_{e-1} \supset L_e = \varpi L_0.$$

The quotients L_i/L_e, $0 \leqslant i \leqslant e$, provide a flag of subspaces of the $\Bbbk_F$-space $L_0/L_e \cong \Bbbk_F^n$. Consequently, $e(\mathcal{L}) \leqslant n$. The lattice chain $\mathcal{L}$ is thus specified by choice of a base point L_0 and a flag of $\Bbbk_F$-subspaces of $L_0/\mathfrak{p}_F L_0 \cong \Bbbk_F^n$.

Given the lattice chain $\mathcal{L} = \{L_i\}$, we set

$$\begin{aligned} \mathfrak{a}(\mathcal{L}) &= \{x \in A : xL_i \subset L_i : i \in \mathbb{Z}\} \\ &= \bigcap_{0 \leqslant i \leqslant e-1} \mathfrak{m}(L_i). \end{aligned}$$

The set $\mathfrak{a}(\mathcal{L})$ is a sub-ring of A, and indeed an $\mathfrak{o}_F$-order in A (such orders are the [illegible] in A). One may describe $\mathfrak{a}(\mathcal{L})$ quite concretely. Let $d_i = \dim_{\Bbbk_F} L_i/L_{i+1}$. We choose a $\Bbbk_F$-basis of L_0/L_e so that the stabilizer of the flag $\{L_i/L_e : 0 \leqslant i < e\}$ is a standard parabolic subgroup of upper triangular block matrices. The inverse image in L_0 of this basis is an $\mathfrak{o}_F$-basis of L_0 (and an F-basis of V). Relative to this basis, $\mathfrak{a}(\mathcal{L})$ becomes identified with the ring of $e \times e$ block matrices $\big(a_{ij}\big)_{0 \leqslant i,j < e}$, in which the ij-block has size $d_i \times d_j$ and entries in $\mathfrak{o}_F$ if $i \leqslant j$, in $\mathfrak{p}_F$ otherwise. Pictorially,

$$\mathfrak{a}(\mathcal{L}) = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F & \dots & \dots & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F & \dots & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F & \dots & \mathfrak{o}_F \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \mathfrak{p}_F & \dots & \dots & \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \dots & \dots & \dots & \mathfrak{p}_F & \mathfrak{o}_F \end{pmatrix}.$$

We may similarly define, for $r \in \mathbb{Z}$,

$$\mathfrak{a}_r(\mathcal{L}) = \{x \in A : xL_i \subset L_{i+r}, 0 \leqslant i < e\}.$$

This is a two-sided $\mathfrak{a}$-module, finitely generated over $\mathfrak{o}_F$. For $r \geqslant 1$, it is an ideal of $\mathfrak{a}$. In particular,

$$\mathfrak{p}_{\mathfrak{a}} = \mathfrak{a}_1(\mathcal{L}) = \begin{pmatrix} \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F & \dots & \dots & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F & \dots & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F & \dots & \mathfrak{o}_F \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \mathfrak{p}_F & \dots & \dots & \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \dots & \dots & \dots & \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix},$$

is the [illegible] of $\mathfrak{a}$: it is a topologically nilpotent ideal of $\mathfrak{a}$ and $\mathfrak{a}/\mathfrak{p}_{\mathfrak{a}} \cong \prod_{0 \leqslant i < e} M_{d_i}(\Bbbk_F)$ is semisimple. We prefer the notation

$$\mathfrak{p}_{\mathfrak{a}}^r = \mathfrak{a}_r(\mathcal{L}), \quad r \in \mathbb{Z},$$

noting that $\mathfrak{p}_{\mathfrak{a}}^r \mathfrak{p}_{\mathfrak{a}}^{-r} = \mathfrak{a}$. We often write $\mathfrak{p}_{\mathfrak{a}} = \operatorname{rad} \mathfrak{a}$.

While we defined $\mathfrak{a}$ in terms of the lattice chain $\mathcal{L}$, the ring $\mathfrak{a}$ determines the lattice chain, up to a change in the numbering:

Proposition. If M is an $\mathfrak{a}$-lattice in V, then $M \in \mathcal{L}$. In particular, $\mathcal{L}$ is the set of all $\mathfrak{a}(\mathcal{L})$-lattices in V.

In this version, the period $e_F(\mathcal{L}) = e$ is given by $\mathfrak{p}_F \mathfrak{a} = \mathfrak{p}_{\mathfrak{a}}^e$ (we therefore write $e = e_F(\mathfrak{a})$ when thinking this way).

1.2.3 Multiplicative structures

Let $\mathfrak{a}$ be a hereditary $\mathfrak{o}_F$-order in A, with Jacobson radical $\mathfrak{p}$ and attached to a lattice chain

$$\mathcal{L} = \{L_i : i \in \mathbb{Z}\}, \quad d_i = \dim_{\Bbbk_F} L_i/L_{i+1}.$$

Let $e = e_F(\mathcal{L})$. We set

$$U_{\mathfrak{a}} = U_{\mathfrak{a}}^0 = \mathfrak{a}^{\times}, \quad U_{\mathfrak{a}}^k = 1+\mathfrak{p}^k, \quad k \geqslant 1.$$

(The groups $U_{\mathfrak{a}}$ are the [illegible] subgroups of G). We will also need the group

$$\begin{aligned} \mathcal{K}_{\mathfrak{a}} &= \{x \in G : x^{-1}\mathfrak{a}x = \mathfrak{a}\} \\ &= \{x \in G : xL \in \mathcal{L},\ L \in \mathcal{L}\} = \operatorname{Aut}_F(\mathcal{L}). \end{aligned}$$

The groups $U_{\mathfrak{a}}^j$ are compact open subgroups of G. The group $\mathcal{K}_{\mathfrak{a}}$ contains $F^{\times}U_{\mathfrak{a}}$, with index $\leqslant e \leqslant n$, and so is compact modulo the centre $F^{\times}$ of G. Observe that each $U_{\mathfrak{a}}^j$, $j \geqslant 0$, is normal in $\mathcal{K}_{\mathfrak{a}}$. Indeed, $\mathcal{K}_{\mathfrak{a}}$ is the G-normalizer of any one of the $U_{\mathfrak{a}}^j$, $j \geqslant 0$.

We have

$$U_{\mathfrak{a}}^0/U_{\mathfrak{a}}^1 \cong \prod_{0 \leqslant i < e} \operatorname{Aut}_{\Bbbk_F}(L_i/L_{i+1}) \cong \prod_{0 \leqslant i < e} \operatorname{GL}_{d_i}(\Bbbk_F),$$

while

$$U_{\mathfrak{a}}^j/U_{\mathfrak{a}}^{j+1} \cong \mathfrak{p}^j/\mathfrak{p}^{j+1}, \quad j \geqslant 1.$$

More generally, we have an isomorphism

$$\begin{aligned} \mathfrak{p}^j/\mathfrak{p}^k &\longrightarrow U_{\mathfrak{a}}^j/U_{\mathfrak{a}}^k, \\ x &\longmapsto 1+x, \end{aligned}$$

whenever $1 \leqslant j < k \leqslant 2j$.

Special case. Say that the lattice chain $\mathcal{L} = \{L_i : i \in \mathbb{Z}\}$ is [illegible] if the parameter $d_i = \dim_{\Bbbk_F} L_i/L_{i+1}$ is constant, independent of i. There then exists $x \in \mathcal{K}_{\mathfrak{a}}$ such that $xL_i = L_{i+1}$, for all i. Indeed, $\mathfrak{p} = x\mathfrak{a} = \mathfrak{a}x$, for any such x. Consequently, one says that the hereditary order attached to a uniform lattice chain is a [illegible]. One may show that the maximal compact mod-centre subgroups of G are the groups $\mathcal{K}_{\mathfrak{a}}$, where $\mathfrak{a}$ ranges over the principal orders in A. The induction theorem of Chapter I suggests that we need to find lots of interesting representations of $\mathcal{K}_{\mathfrak{a}}$, where $\mathfrak{a}$ is a principal order. We approach this in stages.

1.2.4 Duality

We give a simple procedure for constructing characters of the groups $U_{\mathfrak{a}}^j$, $j \geqslant 1$.

Before starting, we recall a fact about local fields. Let $\widehat{F}$ be the group of smooth characters of F. Thus $\widehat{F} \cong F$, as follows. Choose an element $\psi \in \widehat{F}$, $\psi \neq 1$. For $a \in F$, let $a\psi$ denote the character $x \mapsto \psi(ax)$. The map $a \mapsto a\psi$ is then an isomorphism $F \to \widehat{F}$.

We choose a character ψ_F of F, $\psi_F \neq 1$. It is convenient to take ψ_F so that $\mathfrak{o}_F \not\subset \operatorname{Ker} \psi_F \supset \mathfrak{p}_F$: one says that ψ_F [illegible]. Everything we do depends on this choice, but only in a minor way.

We set $\psi_A = \psi_F \circ \operatorname{tr}_A$, where $\operatorname{tr}_A : A \to F$ is the matrix trace. Thus ψ_A is a character of A satisfying $\mathfrak{a} \not\subset \operatorname{Ker} \operatorname{tr}_A \supset \mathfrak{p}$ (as one sees readily from the block matrix picture). For $a \in A$, we may likewise define $a\psi_A \in \widehat{A}$ by $a\psi_A(x) = \psi_A(ax)$, to get an isomorphism $A \cong \widehat{A}$. Here, various elements a may have very different properties, reflected in varying properties of the characters $a\psi_A$.

Proposition. [illegible] $0 \leqslant j < k \leqslant 2j+1$ [illegible] $a \in \mathfrak{p}^{-k}$ [illegible]

$$\psi_{A,a} : 1+x \longmapsto \psi_A(ax), \quad x \in \mathfrak{p}^j,$$

[illegible] $U_{\mathfrak{a}}^{j+1}$ [illegible] $U_{\mathfrak{a}}^{k+1}$ [illegible]

$$\begin{aligned} \mathfrak{p}^{-k}/\mathfrak{p}^{-j} &\longrightarrow \left(U_{\mathfrak{a}}^{j+1}/U_{\mathfrak{a}}^{k+1}\right)^{\wedge}, \\ a &\longmapsto \psi_{A,a}, \end{aligned}$$

[illegible]

The point here is that

$$\mathfrak{p}^{1-m} = \{x \in A : \psi_A(x\mathfrak{p}^m) = \{1\}\},$$

as one sees easily from the block matrix pictures. The algebraic properties of the element a are reflected in the intertwining properties of the character $\psi_{A,a}$. The rôle of ψ_F is straightforward. If ψ'_F is another character of level one, there exists $u \in U_F$ such that $\psi'_F(x) = \psi_F(ux)$, $x \in F$. We then have $\psi_{A,a} = \psi'_{A,u^{-1}a}$.

1.2.5 Strata and intertwining

A [illegible] A is a quadruple $[\mathfrak{a}, k, j, b]$ where

(1) $\mathfrak{a}$ is a hereditary $\mathfrak{o}_F$-order in A, with Jacobson radical $\mathfrak{p}$.

(2) $k > j$ are integers, and

(3) $b \in \mathfrak{p}^{-k}$.

The stratum $[\mathfrak{a}, k, j, b]$ determines the coset $b+\mathfrak{p}^{-j}$. We say two strata are [illegible] if they define the same cosets.

Let $g \in G$; we say that g intertwines the stratum $[\mathfrak{a}, k, j, b]$ if

$$g^{-1}(b+\mathfrak{p}^{-j})g \cap b+\mathfrak{p}^{-j} \neq \varnothing.$$

The point is:

Proposition. [illegible] $[\mathfrak{a}, k, j, b]$ [illegible] A [illegible] $0 \leqslant \left[\dfrac{k}{2}\right] \leqslant j < k$ [illegible] $g \in G$ [illegible] g [illegible] $[\mathfrak{a}, k, j, b]$ [illegible] $\psi_{A,b}$ [illegible] $U_{\mathfrak{a}}^{j+1}$

The proposition applies equally to elements which intertwine one stratum with another, and to the associated characters.

Strata provide a convenient way of describing characters of certain compact open subgroups of G. If π is an irreducible smooth representation of G, we say that π contains the stratum $[\mathfrak{a}, k, j, b]$ if it contains the character $\psi_{A,b}$ of $U_{\mathfrak{a}}^{j+1}/U_{\mathfrak{a}}^{k+1}$.

1.2.6 Field extensions

Let E/F be a finite field extension, of degree d. Let W be an E-vector space of finite dimension m, and write $B = \mathrm{End}_E(W)$. Let $\mathcal{L}$ be an $\mathfrak{o}_E$-lattice chain in W. Thus

$$\mathfrak{b} = \mathfrak{a}_E(\mathcal{L}) = \{x \in B : xL \subset L,\ L \in \mathcal{L}\}$$

is a hereditary $\mathfrak{o}_E$-order in B. However, we may view W as an F-vector space of dimension $n = md$, and then $\mathcal{L}$ provides an $\mathfrak{o}_F$-lattice chain in W and hence a hereditary $\mathfrak{o}_F$-order $\mathfrak{a} = \mathfrak{a}_F(\mathcal{L})$ in $A = \mathrm{End}_F(W)$. We then have, by definition, $\mathfrak{b} = \mathfrak{a} \cap B$ and, more generally,

$$\mathfrak{q}^j = \mathfrak{p}^j \cap B, \quad j \in \mathbb{Z},$$

where $\mathfrak{q} = \mathrm{rad}\ \mathfrak{b}$ and $\mathfrak{p} = \mathrm{rad}\ \mathfrak{a}$ are the respective Jacobson radicals. Further,

$$e_F(\mathcal{L}) = e(E|F)e_E(\mathcal{L}).$$

For $x \in E^\times$ and $L \in \mathcal{L}$, we have $xL \in \mathcal{L}$, so $E^\times \subset \mathcal{K}_{\mathfrak{a}}$.

Looking at it the other way round, we can take a hereditary $\mathfrak{o}_F$-order $\mathfrak{a} = \mathfrak{a}(\mathcal{L})$ in $A = \mathrm{End}_F(V)$, for some finite-dimensional F-vector space V and a field extension E/F inside A. We say that $\mathfrak{a}$ is E [illegible] if $E^\times \subset \mathcal{K}_{\mathfrak{a}}$. This is the same situation as before: the vector space V is an E-space via the inclusion $E \to A$ while $B = \mathrm{End}_E(V)$ is the A-centralizer of E. Every $L \in \mathcal{L}$ is then an $\mathfrak{o}_E$-lattice and $\mathcal{L}$ is an E-lattice chain. Moreover, $\mathfrak{b} = \mathfrak{a} \cap B$ is the hereditary $\mathfrak{o}_E$-order in B attached to $\mathcal{L}$.

Example. There is a particular case worth remembering. We take a field extension E/F of finite degree d, and consider the E-vector space E of dimension one. There is only one $\mathfrak{o}_E$-lattice chain in E, namely $\{\mathfrak{p}_E^j\}_{j\in\mathbb{Z}}$. This gives a principal order $\mathfrak{a}(E)$ in $\mathrm{End}_F(E)$. This is surely E-pure, of F-period $e(E|F)$, and its radical is $\varpi_E\mathfrak{a}(E)$, for any prime element ϖ_E of E. The centralizer of E in $\mathrm{End}_F(E)$ is E itself, and $\mathfrak{a}(E) \cap E = \mathfrak{o}_E$.

1.2.7 Minimal elements

Let E/F be a finite field extension, and suppose $E = F[\alpha]$, $\alpha \in E^\times$. We say that α is [illegible] F if it satisfies two conditions. First, if $\upsilon = \upsilon_E(\alpha)$ and $e = e(E|F)$, then

$$\gcd(\upsilon, e) = 1.$$

To state the second, we choose a prime element ϖ_F of F and form $\alpha_0 = \varpi_F^{-\upsilon}\alpha^e \in U_E$. Let $\tilde{\alpha}_0$ be the image of α_0 in $\Bbbk_E$. The required condition is

$$\Bbbk_E = \Bbbk_F[\tilde{\alpha}_0].$$

Exercise. Take $A = \mathrm{End}_F(V)$ as usual, and let $\alpha \in A$ such that $F[\alpha]$ is a field. Suppose that α is minimal over F. Let $\mathfrak{a}$ be a hereditary order in A such that $\alpha \in \mathcal{K}_{\mathfrak{a}}$. Show that $\mathfrak{a}$ is $F[\alpha]$-pure.

We go back to the hereditary order $\mathfrak{a}(E)$, as in §1.2.6 Example, and write $\mathfrak{p}(E) = \mathrm{rad}\ \mathfrak{a}(E)$.

Proposition 1. [illegible] $\alpha \in E$ [illegible] F [illegible] $E = F[\alpha]$

(1) $\alpha' \in \alpha U^1_{\mathfrak{a}(E)}$ $E' = F[\alpha']$ fi

$$e(E'|F) = e(E|F), \quad f(E'|F) = f(E|F).$$

α' F

(2) $j < k$ Ad α fi $\mathfrak{p}(E)^j/\mathfrak{p}(E)^k$ $\mathfrak{p}^j_E + \mathfrak{p}(E)^k$

These simple facts have some interesting consequences. Set $d = [E{:}F]$ and suppose that $n = -v_E(\alpha) > 0$. If m is an integer, $n > m \geqslant \left[\frac{n}{2}\right]$, then ψ_α defines a character of the group $U^{m+1}_{\mathfrak{a}(E)}$, trivial on $U^{n+1}_{\mathfrak{a}(E)}$. This determines the equivalence class of the stratum $[\mathfrak{a}(E), n, m, \alpha]$.

Proposition 2. $g \in G = \mathrm{Aut}_F(E) \cong \mathrm{GL}_d(F)$ $[\mathfrak{a}(E), n, m, \alpha]$ $g \in E^\times U^{n-m}_{\mathfrak{a}(E)}$ $g \in \mathcal{K}_{\mathfrak{a}(E)}$

Abbreviate $\mathfrak{a} = \mathfrak{a}(E)$, $\mathfrak{p} = \mathfrak{p}(E)$. The intertwining condition is

$$g^{-1}(\alpha + \mathfrak{p}^{-m})g \cap (\alpha + \mathfrak{p}^{-m}) \neq \varnothing.$$

Let t be the greatest integer such that $g \in \mathfrak{p}^t$. If this condition holds, we have

$$\alpha g \alpha^{-1} - g \in \mathfrak{p}^{t+n-m},$$

or, equivalently, $g + \mathfrak{p}^{t+n-m}$ is a fixed point in $\mathfrak{p}^t/\mathfrak{p}^{t+n-m}$ for the action of Adα. That is, $g \in E^\times U^{n-m}_{\mathfrak{a}}$, as required. The converse is immediate. □

Corollary. π $G = \mathrm{GL}_d(F)$ $[\mathfrak{a}, n, n-1, \alpha]$ $n \geqslant 1$:

(a) $\alpha\mathfrak{a} = \mathfrak{p}_a^{-n}$

(b) $F[\alpha]/F$ fi d α F

(c) π ψ_α $U^n_{\mathfrak{a}}$

ρ $\mathcal{K}_{\mathfrak{a}}$ π ψ_α ρ $\pi \cong c\text{-Ind}^G_{\mathcal{K}_{\mathfrak{a}}}\rho$

We can describe, in complete detail, the representations ρ which appear here, but we will do that later.

Before passing on, we briefly consider the case of a stratum $[\mathfrak{a}, n, n-1, \alpha]$, where α is minimal over F, $\alpha\mathfrak{a} = \mathfrak{p}_{\mathfrak{a}}^{-n}$, but $E = F[\alpha]$ is not a maximal subfield of A. Proposition 1(1) fails completely, but an analogue of part (2) holds. In Proposition 2, the intertwining of $[\mathfrak{a}, n, m, \alpha]$ is $U^{n-m}_{\mathfrak{a}} B^\times U^{n-m}_{\mathfrak{a}}$, where B is the A-centralizer of E (but this is not contained in $\mathcal{K}_{\mathfrak{a}}$ unless $F[\alpha]$ is maximal).

1.3 Fundamental strata

We have uncovered a few examples of cuspidal representations constructed by induction. We now turn to the analysis of general (irreducible) representations. The examples we know are not at all typical, but they provide a useful guide.

1.3.1 Fundamental strata

Let $[\mathfrak{a}, n, n{-}1, b]$ be a stratum in $A = \mathrm{End}_F(V)$, where $\dim_F V = N$ say. Writing $\mathfrak{p} = \mathrm{rad}\mathfrak{a}$, we have $b \in \mathfrak{p}^{-n}$ and, more generally, $b^m \in \mathfrak{p}^{-mn}$ for any integer $m \geqslant 1$. We call $[\mathfrak{a}, n, n{-}1, b]$ [illegible] if $b^m \in \mathfrak{p}^{1-mn}$ for some $m \geqslant 1$. Of course, $[\mathfrak{a}, n, n{-}1, b]$ is called [illegible] if it is not non-fundamental.

Proposition. [illegible] $[\mathfrak{a}, n, n{-}1, b]$ [illegible] A [illegible] $\mathfrak{p} = \mathrm{rad}\ \mathfrak{a}$ $e = e(\mathfrak{a})$ [illegible]:

(1) [illegible] $[\mathfrak{a}, n, n{-}1, b]$ [illegible]

(2) [illegible] $\mathfrak{a}_1$ [illegible] A [illegible] e_1 [illegible] $\mathfrak{p}_1$ [illegible] n_1 [illegible]

$$b{+}\mathfrak{p}^{1-n} \subset \mathfrak{p}_1^{-n_1} \quad \textit{and} \quad n_1/e_1 < n/e.$$

This has useful consequences.

Corollary. (1) [illegible] $[\mathfrak{a}, n, n{-}1, b]$ [illegible] A [illegible] $[\mathfrak{a}', n', n'{-}1, 0]$ [illegible]

$$(n'{-}1)/e' \geqslant n/e,$$

[illegible] $e = e(\mathfrak{a})$ $e' = e(\mathfrak{a}')$

(2) [illegible] $i = 1, 2$ [illegible] $[\mathfrak{a}_i, n_i, n_i{-}1, b_i]$ [illegible] A [illegible]

$$n_1/e(\mathfrak{a}_1) = n_2/e(\mathfrak{a}_2).$$

1.3.2 Application to representations

Let π be an irreducible smooth representation of $G = \mathrm{Aut}_F(V)$. Say that π [illegible] if there is a maximal order $\mathfrak{m}$ in $A = \mathrm{End}_F(V)$ such that π admits a $U^1_{\mathfrak{m}}$-fixed point.

This condition, we note, does not depend on the maximal order $\mathfrak{m}$, since any two choices of $\mathfrak{m}$ are G-conjugate (If $\mathfrak{a}$ is a hereditary order in A, contained in some maximal order $\mathfrak{m}$, then $U^1_{\mathfrak{m}} \subset U^1_{\mathfrak{a}}$. This explains why we need only consider maximal orders here). In principle, we already know everything about cuspidal representations of level zero (1.1.5 Corollary), so we concentrate on the case of [illegible]

So, let π be an irreducible smooth representation of G, not of level zero. We define the [illegible] of π to be

$$\ell(\pi) = \min n/e(\mathfrak{a}),$$

where $(n, \mathfrak{a})$ ranges over all pairs consisting of a hereditary order $\mathfrak{a}$ in A and an integer $n \geqslant 1$ such that π contains the trivial character of $U_{\mathfrak{a}}^{n+1}$. Note that $e(\mathfrak{a}) \leqslant \dim V$, so $\ell(\pi) \geqslant 1/\dim V$.

Theorem. π G $[\mathfrak{a}, n, n-1, b]$ π $n/e(\mathfrak{a}) = \ell(\pi)$ π

Let $[\mathfrak{a}, n, n-1, b]$ be a stratum contained in π. If it is not fundamental, we apply the proposition above to get a hereditary order $\mathfrak{a}_1$ and an integer n_1 such that $n_1/e_1 < n/e$ and $b+\mathfrak{p}^{1-n} \subset \mathfrak{p}_1^{-n_1}$ (using $e_1 = e(\mathfrak{a}_1)$ and so on). The containment implies $\mathfrak{p}^{1-n} \subset \mathfrak{p}_1^{-n_1}$ so, dualizing, we get $\mathfrak{p}_1^{1+n_1} \subset \mathfrak{p}^n$. The representation π thus contains the character $\psi_{A,b}$ restricted to $U_{\mathfrak{a}_1}^{1+n_1}$, and this character is trivial again by the containment. □

So, we start analyzing the irreducible representations of G in terms of the fundamental strata they contain.

1.3.3 The characteristic polynomial

We choose a prime element ϖ_F of F: we only use this for comparison purposes, so the choice is irrelevant.

We are given a stratum $[\mathfrak{a}, n, n-1, b]$ in A. Let $e = e(\mathfrak{a})$, and let $g = \gcd(n, e)$. The element $b_0 = \varpi_F^{n/g} b^{e/g}$ then lies in $\mathfrak{a}$. If $\mathfrak{a} = \mathfrak{a}(\mathcal{L})$, for a lattice chain $\mathcal{L} = \{L_i\}_{i \in \mathbb{Z}}$, the element b_0 defines an endomorphism of L_0/L_e. Let $\varphi_b(t) \in \Bbbk_F[t]$ be the characteristic polynomial of this endomorphism (which is independent of the choice of base point L_0). In particular, $\varphi_b(t)$ is monic, of degree $N = \dim_F V$. Notice that $\varphi_b(t)$ comes as a product of e polynomials, since L_0/L_e comes equipped with a flag L_i/L_e of b_0-stable subspaces.

Proposition. (1) $[\mathfrak{a}, n, n-1, b]$ $\varphi_b(t) \neq t^N$

(2) $i = 1, 2$ $[\mathfrak{a}_i, n_i, n_i-1, b_i]$ A $\varphi_{b_1} = \varphi_{b_2}$

In particular, if two fundamental strata occur in the same irreducible representation of G, they share the same characteristic polynomial. This expresses a key dichotomy.

Theorem. π G $\varphi(t)$ π $\varphi(t)$ distinct π

This is proved by producing explicitly a non-zero Jacquet module—we shall say no more about it.

A fundamental stratum whose characteristic polynomial has at least two distinct irreducible factors will be called [illegible] The theorem reduces us to the case where this "characteristic polynomial of π" is a power of an irreducible polynomial $f(t)$ with $f(t) \neq t$.

1.3.4 Nonsplit fundamental strata

These have a very useful property.

Lemma. [illegible] $[\mathfrak{a}, n, n-1, b]$ [illegible] b [illegible] $\mathcal{K}_{\mathfrak{a}}$

[illegible] Let $\mathfrak{p} = \operatorname{rad}\mathfrak{a}$ and let $\{L_i : i \in \mathbb{Z}\}$ be the lattice chain defining $\mathfrak{a}$. Since b lies in $\mathfrak{p}^{-n}$ but not in $\mathfrak{p}^{1-n}$, the condition $b \in \mathcal{K}_{\mathfrak{a}}$ is equivalent to $bL_i = L_{i-n}$, for all i. Assuming $n \geqslant 0$, we have $b^j L_i \subset L_{i-jn}$, $j \geqslant 1$. One of these containments is strict if and only if the characteristic polynomial of the stratum is divisible by t. $\square$

Definition. [illegible] $\mathfrak{a}$ [illegible] A [illegible] $\mathfrak{p}$ [illegible] $[\mathfrak{a}, n, n-1, \alpha]$ [illegible] simple [illegible]

(1) $\alpha\mathfrak{a} = \mathfrak{p}^{-n}$

(2) [illegible] $F[\alpha]$ [illegible]

(3) α [illegible] F

In particular, a simple stratum is non-split fundamental.

Theorem. [illegible] π [illegible] $G \cong \mathrm{GL}_N(F)$ [illegible] π [illegible] $[\mathfrak{a}, n, n-1, \alpha]$

[illegible] Let $[\mathfrak{a}, n, n-1, b]$ be a nonsplit fundamental stratum occurring in π. Let $g = \gcd(n, e(\mathfrak{a}))$. Let $\mathcal{L}$ be the lattice chain defining $\mathfrak{a}$. Every lattice $b^i \varpi_F^j L_0$ lies in $\mathcal{L}$ and this set of lattices is a lattice chain $\mathcal{L}_0$, contained in $\mathcal{L}$, of period $e(\mathfrak{a})/g$. Let $\mathfrak{a}_0 = \mathfrak{a}(\mathcal{L}_0)$. We have $U_{\mathfrak{a}_0}^{n/g} \subset U_{\mathfrak{a}}^n$, so π contains the character $\psi_{A,b}|_{U_{\mathfrak{a}_0}^{n/g}}$. That is, π contains the (nonsplit fundamental) stratum $[\mathfrak{a}_0, n/g, n/g-1, b]$. In other words, we might as well assume that $e(\mathfrak{a})$ is relatively prime to n.

We form the element $b_0 = \varpi_F^n b^{e(\mathfrak{a})} \in U_{\mathfrak{a}}$. This acts on L_0/L_1 as an automorphism with characteristic polynomial $\varphi_b(t) = \tilde{f}(t)^r$, where $\tilde{f}[t] \in \Bbbk_F[t]$ is irreducible and r is some positive integer. We choose a maximal flag of b_0-invariant subspaces of L_0/L_1. Applying powers of b, we get such a flag in each L_i/L_{i+1}. Putting these together, we get a lattice chain $\mathcal{L}'$ containing $\mathcal{L}$, of period $re(\mathfrak{a})$, invariant under b, and such that b_0 acts on each L'_j/L'_{j+1} as an automorphism with characteristic polynomial $\tilde{f}(t)$. Let $\mathfrak{a}' = \mathfrak{a}(\mathcal{L}')$.

We choose a monic polynomial $f[t] \in \mathfrak{o}_F[t]$ reducing to $\tilde{f}[t]$. Using a block matrix picture, we find $\alpha_0 \in U_{\mathfrak{a}}$ with minimal polynomial $f[t]$ and acting on each L'_i/L'_{i+1} in the same way as b_0 (the element α_0 is represented by a diagonal block matrix, in which all diagonal blocks are the same). We have $\alpha_0 \equiv b_0 \pmod{U^1_{\mathfrak{a}}}$. We work back to get $\alpha \in \mathcal{K}_{\mathfrak{a}}$, with $\alpha_0 = \varpi_F^n \alpha^{e(\mathfrak{a})}$ and $\alpha \equiv b \pmod{U^1_{\mathfrak{a}_1}}$. The stratum $[\mathfrak{a}_1, n_1, n_1{-}1, \alpha]$, where $n_1 = rn$, is then simple. However, $U^{n_1}_{\mathfrak{a}_1} \subset U^n_{\mathfrak{a}}$, so π contains the character $\psi_{A,b} = \psi_{A,\alpha}$ of $U^{n_1}_{\mathfrak{a}_1}$. $\square$

1.4 Prime dimension

Let l be a prime number, and let $G = \mathrm{GL}_l(F)$. Let π be an irreducible cuspidal representation of G, which is not of level zero.

1.4.1 A trivial case

We know that π must contain a non-split fundamental stratum, and hence a simple stratum $[\mathfrak{a}, n, n{-}1, \alpha]$. The field $F[\alpha]/F$ is a subfield of $M_l(F)$, so the degree $\big[F[\alpha]{:}F\big]$ is either 1 or l.

Consider first the case $\big[F[\alpha]{:}F\big] = 1$, that is, $\alpha \in F^\times$. Since $\alpha\mathfrak{a} = \mathfrak{p}^{-n}$, where $\mathfrak{p} = \mathrm{rad}\,\mathfrak{a}$, the integer n is divisible by $e = e(\mathfrak{a})$ and so $\upsilon_F(\alpha) = -n/e$. There exists a character χ of $F^\times$ such that $\chi|_{U_F^{n/e}} = \psi_{F,\alpha}$. We then have

$$\chi \circ \det|_{U^n_{\mathfrak{a}}} = \psi_{A,\alpha}.$$

The representation $\chi^{-1}\pi : g \mapsto \chi(\det g)^{-1}\pi(g)$ has a $U^n_{\mathfrak{a}}$-fixed point, whence $\ell(\chi^{-1}\pi) < n/e = \ell(\pi)$. We have so reduced the level and can start again.

1.4.2 The general case

We consider irreducible cuspidal representations π of $G = \mathrm{GL}_l(F)$ such that $0 < \ell(\pi) \leqslant \ell(\chi\pi)$, for any character χ of $F^\times$. Such representations π are said to be "of minimal positive level".

It will be useful to take a more systematic approach. Let $\left[\mathfrak{a}, n, \left[\frac{n}{2}\right], \alpha\right]$ be a stratum in $A = M_l(F)$, such that $[\mathfrak{a}, n, n{-}1, \alpha]$ is simple and $\alpha \notin F$. Set $E = F[\alpha]$. Since E is a maximal subfield of A, $\mathfrak{a}$ is the unique E-pure hereditary order in A. It is principal, with $e(\mathfrak{a}) = e(E|F)$.

We form the groups

$$H^1 = H^1(\alpha, \mathfrak{a}) = U^1_E U^{[n/2]+1}_{\mathfrak{a}},$$
$$J^1 = J^1(\alpha, \mathfrak{a}) = U^1_E U^{[n+1/2]}_{\mathfrak{a}}.$$

We define a set $\mathcal{C}(\mathfrak{a},\alpha,\psi_F)=\mathcal{C}(\alpha)$ of characters of H^1 by the condition $\theta\in\mathcal{C}(\alpha)$ if $\theta|_{U_{\mathfrak{a}}^{[n/2]+1}}=\psi_\alpha$. The characters $\theta\in\mathcal{C}(\alpha)$ are the [illegible] defined by α. An element $g\in G$ intertwines $\theta\in\mathcal{C}(\alpha)$ if and only if $g\in E^\times J^1(\alpha)$ (see Subsection 1.2.7). We also remark that the group $H^1(\alpha)$ and the set $\mathcal{C}(\mathfrak{a},\alpha)$ only determine α modulo $\mathfrak{p}_{\mathfrak{a}}^{-[n/2]}$. They do not determine the field $F[\alpha]$ in general.

Theorem. [illegible] $\left[\mathfrak{a},n,\left[\frac{n}{2}\right],\alpha\right]$ [illegible] $n\geqslant 1$ [illegible] $[\mathfrak{a},n,n-1,\alpha]$ [illegible] $F[\alpha]\neq F$

(1) [illegible] $\theta\in\mathcal{C}(\alpha)$ [illegible] G [illegible] θ [illegible]

(2) [illegible] $\theta_1,\theta_2\in\mathcal{C}(\alpha)$ [illegible] θ_1 [illegible] θ_2 [illegible] G [illegible] $\theta_1=\theta_2$

(3) [illegible] π [illegible] G [illegible] $[\mathfrak{a},n,n-1,\alpha]$ [illegible] π [illegible] $\theta\in\mathcal{C}(\alpha)$

[illegible] Part (1) is already done. In (2), any $g\in G$ which intertwines θ_1 with θ_2 must intertwine ψ_α (on $U_{\mathfrak{a}}^{[n/2]+1}$) with itself. Therefore $g\in E^\times J^1(\alpha)$ and so $\theta_1^g=\theta_1$.

In part (3), we know from II §4 that π contains a simple stratum $[\mathfrak{a},n,n-1,\alpha]$, with $\alpha\notin F$. It therefore contains a stratum $\left[\mathfrak{a},n,\left[\frac{n}{2}\right],\beta\right]$, with $\beta\equiv\alpha \pmod{U_{\mathfrak{a}}^1}$. We can change our choice of α and assume $\beta=\alpha$. Thus π contains an irreducible representation of $H^1(\alpha)$ containing ψ_α (on $U_{\mathfrak{a}}^{[n/2]+1}$). The only such representations are the simple characters $\theta\in\mathcal{C}(\alpha)$. ☐

Parts (2), (3) of Theorem 1 give a sort of uniqueness. There is stronger version.

Proposition. [illegible] $i=1,2$ [illegible] $[\mathfrak{a},n_i,n_i-1,\alpha_i]$ [illegible] $\theta_i\in\mathcal{C}(\mathfrak{a},\alpha_i)$ [illegible] θ_1 [illegible] θ_2 [illegible] G [illegible] $g\in\mathcal{K}_{\mathfrak{a}}$ [illegible] $\theta_2=\theta_1^g$ [illegible] $\alpha_1^g\equiv\alpha_2 \pmod{\mathfrak{p}^{-[n/2]}}$

[illegible] If the θ_i intertwine, so do the simple strata $[\mathfrak{a},n_i,n_i-1,\alpha_i]$. It follows that $n_1=n_2=n$ say, and that the α_i have the same characteristic polynomials. They are therefore conjugate modulo $\mathfrak{p}^{1-n}$ (we can write down an explicit matrix representative directly from the characteristic polynomial). One proceeds by successive approximation, using the fixed point Theorem 1.2.7 Proposition 1. This reduces us to the case $\alpha_1\equiv\alpha_2 \pmod{\mathfrak{p}^{-[n/2]}}$ and we are done. ☐

1.4.3 The inducing representation

We continue in the same situation, with a simple character $\theta\in\mathcal{C}(\alpha)$. The G-normalizer of θ is the group $\boldsymbol{J}(\alpha)=E^\times J^1(\alpha)$, and this is also the set of $g\in G$ which intertwine θ. We need to know, therefore, the irreducible representations of $\boldsymbol{J}$ containing θ. We start with the group J^1.

Proposition. [illegible] $\eta = \eta_\theta$ [illegible] J^1 [illegible] $\eta|_{H^1}$ [illegible] θ [illegible] $\eta|_{H^1}$ [illegible] θ

[illegible] This is based on a standard construction from the representation theory of finite p-groups. Let p be the characteristic of the field $\Bbbk_F$.

Let $Z = H^1/\mathrm{Ker}\,\theta$, $Y = J^1/\mathrm{Ker}\,\theta$. Thus Z is a finite, cyclic p-group, Z is central in Y and Y/Z is a finite, elementary abelian p-group. To prove the result, we have to show that Z is the centre of Y. This is done as follows. If we take $x, y \in J^1$, the commutator $[x,y] = xyx^{-1}y^{-1}$ lies in H^1 and the pairing $(x,y) \mapsto \theta[x,y]$ is an alternating form on the quotient ($\mathbb{F}_p$-vector space) J^1/H^1. We have to show that this form is nondegenerate.

It is enough to take elements $1{+}x, 1{+}y \in U_{\mathfrak{a}}^{[n+1/2]}$ in which case

$$\theta[1{+}x, 1{+}y] = \psi_A(\alpha(xy{-}yx)) = \psi_F \mathrm{tr}_A(\alpha xy - \alpha yx) = \psi_A(y(\alpha x - x\alpha)).$$

This vanishes for all y if and only if $\alpha x{-}x\alpha \in \mathfrak{p}_{\mathfrak{a}}^{1-[n+1/2]}$, which comes down to $1{+}x \in U_E^{[n+1/2]} U_{\mathfrak{a}}^{[n/2]+1} \subset H^1$. □

Theorem. [illegible] Λ [illegible] $\boldsymbol{J}(\alpha)$ [illegible] θ [illegible] $\Lambda|_{J^1(\alpha)}$ [illegible] η_θ [illegible] Λ' [illegible] $\boldsymbol{J}(\alpha)$ [illegible] θ [illegible] Λ' [illegible] Λ [illegible] $\Lambda' \cong \Lambda$

1.4.4 Uniqueness

There is one more uniqueness issue to be resolved. If π is any irreducible smooth representation of $\mathrm{GL}_n(F)$, let $D(\pi)$ be the group of [illegible] characters χ of $F^\times$ such that $\chi\pi \cong \pi$. Thus $D(\pi)$ is cyclic, of order $d(\pi)$ dividing n.

In the case to hand, where π is a cuspidal representation of $\mathrm{GL}_l(F)$ of minimal positive level, $d(\pi)$ is 1 or l. Looking at the inducing representation theorem, we get:

Lemma. [illegible] π [illegible] G [illegible] $[\mathfrak{a}, n, n{-}1, \alpha]$ [illegible] π [illegible] $d(\pi)e(\mathfrak{a}) = l$

1.4.5 Summary

Let $[\mathfrak{a}, n, n{-}1, \alpha]$ be a simple stratum in $A = M_l(F)$, with $n \geqslant 1$ and $\alpha \notin F$. Let $\theta \in \mathcal{C}(\alpha)$. An [illegible] θ is an irreducible representation of $\boldsymbol{J}(\alpha)$ which contains θ.

Classification Theorem. [illegible] (π, V) [illegible] $G = \mathrm{GL}_l(F)$ [illegible]

(1) [illegible] $[\mathfrak{a}, n, n{-}1, \alpha]$ [illegible] $M_l(F)$ [illegible] π [illegible] $\theta \in \mathcal{C}(\alpha)$ [illegible] G [illegible] θ [illegible] π

(2) Λ θ π fi

$$\pi \cong c\text{-}\mathrm{Ind}_{\boldsymbol{J}(\alpha)}^{G}\Lambda.$$

Λ $\boldsymbol{J}(\alpha)$ θ V^θ V

With suitable definitions (to come), this is very close to the final classification theorem for general dimension N.

1.5 Simple strata and simple characters

We define the general notion of simple character. A simple character has to be defined via a simple stratum, but one cannot usually retrieve the simple stratum from the simple character. This ambiguity has some trivial components, but others seem to be of arithmetic significance: the picture is not properly understood at this stage.

Unless explicitly stated otherwise, V is an F-vector space of finite dimension N, and $A = \mathrm{End}_F(V)$, $G = \mathrm{Aut}_F(V)$. We will give hardly any proofs, as these require a quite elaborate calculus and many pages. The material is all available in the literature, especially [5].

1.5.1 Adjoint map

We record a technicality for later use.

Let E/F be a subfield of A, and let B denote the A-centralizer of E. The vector space A carries the nondegenerate, symmetric bilinear form $(x,y) \mapsto \mathrm{tr}_A(xy)$ induced by the matrix trace $\mathrm{tr} : A \to F$. Let C temporarily denote the orthogonal complement of B with respect to this form. If we have an element $\beta \in E$ such that $E = F[\beta]$, then surely C contains the space $\{\beta x - x\beta : x \in A\}$. The kernel of the map $x \mapsto a_\beta(x) = \beta x - x\beta$ is B so, comparing dimensions, we get

$$C = a_\beta(A).$$

Note that, if E/F is inseparable, then $B \subset C$. In general, the space C is a (B,B)-bimodule.

Proposition 1. (1) (B,B) $A \to B$

(2) E $\mathrm{Hom}_{(B,B)}AB$ $f : A \to B$ C fi $f(E) \subset E$

Part (2) is formal and we give no proof. To prove (1), we choose non-trivial characters $\psi_F \in \widehat{F}$, $\psi_E \in \widehat{E}$ and set $\psi_A = \psi_F \circ \mathrm{tr}_A$, $\psi_B = \psi_E \circ \mathrm{tr}_B$. For $a \in A$, let $a\psi_A \in \widehat{A}$ denote the character $x \mapsto \psi_A(ax)$, and similarly for B. The restriction

$a\psi_A|_B$ lies in $\widehat{B}$, and there is a unique $b \in B$ such that $a\psi_A|_B = b\psi_B$. One may choose $a\psi_A$ to have non-trivial restriction to B, and $a \mapsto b$ is the desired map. $\square$

In the context of the last proof, we take ψ_F, ψ_E both [illegible] Together, they give a non-trivial (B, B)-bimodule homomorphism $s : A \to B$. Any such map we call a [illegible] on A relative to E/F. Observe that if s' is some other tame corestriction, there exists $u \in U_E$ such that $s' = us$. Note also that, if $E = F[\beta]$, we have an infinite exact sequence

$$\cdots \xrightarrow{s} A \xrightarrow{a_\beta} A \xrightarrow{s} A \xrightarrow{a_\beta} A \xrightarrow{s} \cdots$$

We shall be concerned with the behaviour of tame corestriction relative to hereditary orders.

Proposition 2. [illegible] E/F [illegible] A [illegible] $\mathfrak{a}$ [illegible] E [illegible] A [illegible] $\mathfrak{p} = \mathrm{rad}\ \mathfrak{a}$ [illegible] B [illegible] A [illegible] E [illegible] $\mathfrak{b} = \mathfrak{a} \cap B$ $\mathfrak{q} = \mathfrak{p} \cap B = \mathrm{rad}\ \mathfrak{b}$ [illegible] s [illegible] A [illegible] E/F [illegible]

$$s(\mathfrak{p}^j) = \mathfrak{q}^j, \quad j \in \mathbb{Z}.$$

1.5.2 Critical exponent

Two strata $[\mathfrak{a}, n, m, b_i]$ in A will be deemed equivalent if $b_1 \equiv b_2 \pmod{\mathfrak{p}^{-m}}$, where $\mathfrak{p} = \mathrm{rad}\ \mathfrak{a}$.

A stratum $[\mathfrak{a}, n, m, a]$ is [illegible] if $a\mathfrak{a} = \mathfrak{p}^{-n}$, $F[a]$ is a field and $\mathfrak{a}$ is $F[a]$-pure. So, let $[\mathfrak{a}, n, m, \beta]$ be a pure stratum in A, and write $E = F[\beta]$. Let B denote the A-centralizer of E, put $\mathfrak{b} = \mathfrak{a} \cap B$ and $\mathfrak{q} = \mathfrak{p} \cap B$. We define

$$\mathfrak{n}_k = \{x \in \mathfrak{a} : a_\beta(x) \in \mathfrak{p}^k\}, \quad k \in \mathbb{Z}.$$

For $k \leqslant -n$, $\mathfrak{n}_k = \mathfrak{a}$, while $\mathfrak{b} \subset \mathfrak{n}_k$ for all k. The intersection $\bigcap_{k \in \mathbb{Z}} \mathfrak{n}_k$ is $\mathfrak{b}$, so $\mathfrak{n}_k \subset \mathfrak{b}+\mathfrak{p}$ for all sufficiently large k. Of course, if $E = F$, then $\mathfrak{n}_k = \mathfrak{a} = \mathfrak{b}$ for all k. Ignoring this case for the moment, we define the [illegible]

$$k_0(\beta, \mathfrak{a}) = \max\{k \in \mathbb{Z} : \mathfrak{n}_k \not\subset \mathfrak{b}+\mathfrak{p}\}.$$

When $E = F$, it works best to set $k_0(\beta, \mathfrak{a}) = -\infty$ (but we often forget this exceptional case). Otherwise, we have $k_0(\beta, \mathfrak{a}) \geqslant -n$.

The dependence of $k_0(\beta, \mathfrak{a})$ on $\mathfrak{a}$ is straightforward.

Proposition 1. [illegible] V' [illegible] E [illegible] $[\mathfrak{a}', n', m', \beta]$ [illegible] $A' = \mathrm{End}_F(V')$ [illegible]

$$\frac{n'}{n} = \frac{e(\mathfrak{a}')}{e(\mathfrak{a})} = \frac{k_0(\beta, \mathfrak{a}')}{k_0(\beta, \mathfrak{a})}.$$

In particular, the quantity $k_0(\beta) = k_0(\beta, \mathfrak{a}(E))$ depends only on β and determines all the other values $k_0(\beta, \mathfrak{a})$.